

THE Q -SHAPED DERIVED CATEGORY OF A RING — COMPACT AND PERFECT OBJECTS

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ABSTRACT. A chain complex can be viewed as a representation of a certain self-injective quiver with relations, Q . To define Q , include a vertex q_n and an arrow $q_n \xrightarrow{\partial} q_{n-1}$ for each integer n . The relations are $\partial^2 = 0$.

Replacing Q by a general self-injective quiver with relations, it turns out that some of the key properties of chain complexes generalise. Indeed, consider the representations of such a Q with values in ${}_A\text{Mod}$ where A is a ring. We showed in earlier work that these representations form the objects of the Q -shaped derived category, $\mathcal{D}_Q(A)$, which is triangulated and generalises the classic derived category $\mathcal{D}(A)$. This follows ideas of Iyama and Minamoto.

While $\mathcal{D}_Q(A)$ has many good properties, it can also diverge dramatically from $\mathcal{D}(A)$. For instance, let Q be the quiver with one vertex q , one loop ∂ , and the relation $\partial^2 = 0$. By analogy with perfect complexes in the classic derived category, one may expect that a representation with a finitely generated free module placed at q is a compact object of $\mathcal{D}_Q(A)$, but we will show that this is, in general, false.

The purpose of this paper, then, is to compare and contrast $\mathcal{D}_Q(A)$ and $\mathcal{D}(A)$ by investigating several key classes of objects: Perfect and strictly perfect, compact, fibrant, and cofibrant.

1. INTRODUCTION

A chain complex over a ring A can be viewed as an ${}_A\text{Mod}$ -valued representation of the quiver

$$(\#1) \quad \Gamma = \cdots \xrightarrow{\partial} \bullet \xrightarrow{\partial} \bullet \xrightarrow{\partial} \bullet \xrightarrow{\partial} \cdots$$

with the relations $\partial^2 = 0$. Expressed more formally, the category of chain complexes $\text{Ch } A$ can be identified with ${}_{Q, A}\text{Mod}$, the category of additive functors $Q \rightarrow {}_A\text{Mod}$, where Q is the path category of Γ with these relations.

A key property of Γ is that it is a *self-injective* quiver with relations; equivalently, Q has a Serre functor (see 2.6). Following ideas of Iyama and Minamoto [24], [25], we showed in [18] that many properties of chain complexes can be generalised to ${}_{Q, A}\text{Mod}$ when Q is another suitable category with a Serre functor. This permits Q to be the path category of many different quivers with relations. Among them

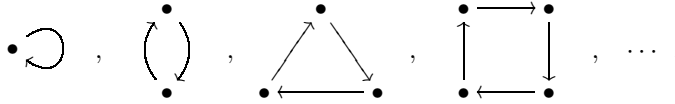
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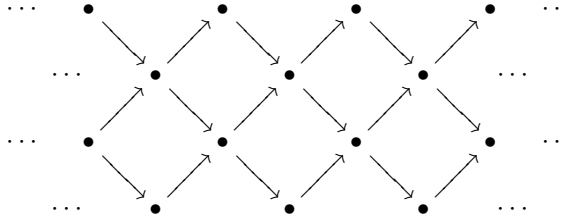
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are cyclic quivers, like:



repetitive quivers, like:



and double quivers, like:



To wit, the following was proved in [18] when Q has a Serre functor.

- There is a notion of “quasi-isomorphism” in $Q, A \text{ Mod}$, which generalises the usual notion of quasi-isomorphism in $\text{Ch } A$. Inverting the quasi-isomorphisms in $Q, A \text{ Mod}$ gives a category $\mathcal{D}_Q(A)$, which we call the Q -shaped derived category of A .
- $\mathcal{D}_Q(A)$ is a triangulated category. If Q is the path category of $(\#1)$ with the relations $\partial^2 = 0$, then $\mathcal{D}_Q(A)$ can be identified with the classic derived category $\mathcal{D}(A)$.
- $Q, A \text{ Mod}$ contains three classes of objects, \mathcal{E} , ${}^\perp\mathcal{E}$, and \mathcal{E}^\perp , which generalise the exact, semi-projective, and semi-injective complexes (see Example 3.3).
- These classes provide $Q, A \text{ Mod}$ with two model category structures, which generalise the standard projective and injective model category structures on $\text{Ch } A$ (see 2.10).
- Either model category structure on $Q, A \text{ Mod}$ has the associated homotopy category $\text{Ho}(Q, A \text{ Mod}) = \mathcal{D}_Q(A)$.

However, there are key properties of the (classic) derived category $\mathcal{D}(A)$ which do not generalise to $\mathcal{D}_Q(A)$, and the purpose of this paper is to compare and contrast these categories by investigating several key classes of objects.

1.1 (Strictly perfect objects). *Recall that a strictly perfect complex P is a bounded complex of finitely generated projective modules. It is well-known that a strictly perfect complex is compact in $\mathcal{D}(A)$, that is, $\text{Hom}_{\mathcal{D}(A)}(P, -)$ preserves set-indexed coproducts. However, as shown in Theorem A.3, the corresponding property does not always hold for $\mathcal{D}_Q(A)$.*

Theorem A. *Let Q be the path category of the Jordan quiver*



with the relation $\partial^2 = 0$. There exists a commutative noetherian ring A and an object

$$M = A \overset{x}{\curvearrowright}$$

in $\mathcal{D}_Q(A)$ which is not compact in $\mathcal{D}_Q(A)$, despite having the finitely generated projective module A at the (unique) vertex.

Note that for this quiver, ${}_{Q,A}\text{Mod}$ is the category of *differential A -modules*, i.e. A -modules equipped with an endomorphism squaring to zero. Differential modules have been studied intensively in the literature, see for instance [1], but Theorem A is new.

The object M in Theorem A is an obvious generalization of a strictly perfect complex. Indeed, we will say that a functor $K : Q \rightarrow {}_A\text{Mod}$ in ${}_{Q,A}\text{Mod}$ is *strictly perfect* if it has finite support, that is, if the set $\text{supp } K = \{q \in Q \mid K(q) \neq 0\}$ is finite, and the A -module $K(q)$ is finitely generated and projective for each $q \in Q$ (see Definition 5.3). As shown by Theorem A, strictly perfect objects need not be compact in $\mathcal{D}_Q(A)$, but we do have the following (positive) result, which is proved in Section 5. For the notion of *cycles* in the category Q , see Definition 5.13 under the assumptions in Setup 2.9.

Theorem B. *Assume Setup 2.9 and one of the next conditions:*

- (1) *The category Q has no cycles, or*
- (2) *The ring A has finite left global dimension.*

The following assertions hold:

- (a) *Every strictly perfect object is compact in $\mathcal{D}_Q(A)$.*
- (b) *Every strictly perfect object is cofibrant in the projective model structure on ${}_{Q,A}\text{Mod}$; that is, every strictly perfect object is in ${}^\perp\mathcal{E}$ (see Definition 5.3).*
- (c) *An object in $\mathcal{D}_Q(A)$ is perfect if and only if it is isomorphic to a strictly perfect object.*

Notice that in the special case of chain complexes, the category Q has no cycles and hence condition (1) in Theorem B holds. Also note that by condition (2) in Theorem B, the ring in Theorem A must necessarily have infinite global dimension (and, indeed, it has). We now turn to perfect objects in $\mathcal{D}_Q(A)$, which are already mentioned in part (c) of Theorem B.

1.2 (Perfect objects). *Since strictly perfect objects in the category ${}_{Q,A}\text{Mod}$ need not be compact in $\mathcal{D}_Q(A)$, we propose in 5.3 the following definition:*

$$X \in \mathcal{D}_Q(A) \text{ is perfect} \iff \left\{ \begin{array}{l} \text{There is an isomorphism } X \cong K \text{ in } \mathcal{D}_Q(A) \\ \text{for some strictly perfect object } K \in {}^\perp\mathcal{E}. \end{array} \right.$$

Observe that if condition (1) or (2) in Theorem B holds, then by part (c) of that theorem, the perfect objects are just those which are isomorphic in $\mathcal{D}_Q(A)$ to a strictly perfect object.

With this definition, we show in Section 5 the following result where $\mathcal{D}_Q^{\text{perf}}(A)$ is the class of perfect objects and $\mathcal{D}_Q(A)^c$ is the class of compact objects in $\mathcal{D}_Q(A)$:

Theorem C. *Assume Setup 2.9. There is an inclusion of triangulated subcategories of the Q -shaped derived category of A ,*

$$\mathcal{D}_Q^{\text{perf}}(A) \subseteq \mathcal{D}_Q(A)^c.$$

Equality holds if and only if the subcategory $\mathcal{D}_Q^{\text{perf}}(A)$ is thick.

A triangulated subcategory of a triangulated category is called *thick* if it is closed under direct summands (see also Definition 4.3). In the special case of chain complexes, it is known that $\mathcal{D}^{\text{perf}}(A)$ is a thick subcategory of $\mathcal{D}(A)$; whence equality holds in Theorem C. We conjecture that the same is true in general:

Conjecture. *Assume Setup 2.9. There is always an equality of triangulated subcategories of the Q -shaped derived category of A ,*

$$\mathcal{D}_Q^{\text{perf}}(A) = \mathcal{D}_Q(A)^c.$$

1.3 (Compact objects). *The notion of (strictly) perfect objects aside, another main result is the following, generalizing that the (classic) derived category is compactly generated. This will be proved in Section 4 (the objects in the set \mathbb{S} are defined in 2.8):*

Theorem D. *Assume Setup 2.9. Then $\mathcal{D}_Q(A)$ is a compactly generated triangulated category; in fact,*

$$\mathbb{S} = \{S_q(A) \mid q \in Q\}$$

is a set of compact generators for $\mathcal{D}_Q(A)$. In particular, one has

$$\mathcal{D}_Q(A) = \text{Loc}(\mathbb{S}) \quad \text{and} \quad \mathcal{D}_Q(A)^c = \text{Thick}(\mathbb{S}).$$

1.4 (Cofibrant and fibrant objects). *Our final main result concerns the two model category structures on the functor category ${}_Q, A\text{Mod}$. In the projective model structure on ${}_Q, A\text{Mod}$, the class of cofibrant objects is ${}^\perp\mathcal{E}$ and in the injective model structure the class of fibrant objects is \mathcal{E}^\perp , see [18, Thm. 6.1]. The following shows that over a ring with finite global dimension, it is possible to give hands-on descriptions of these classes:*

Theorem E. *Assume Setup 2.9. There are inclusions:*

$$\begin{aligned} {}^\perp\mathcal{E} &\subseteq \{X \in {}_Q, A\text{Mod} \mid X(q) \in {}_A\text{Prj} \text{ for all } q \in Q\} \quad \text{and} \\ \mathcal{E}^\perp &\subseteq \{X \in {}_Q, A\text{Mod} \mid X(q) \in {}_A\text{Inj} \text{ for all } q \in Q\}, \end{aligned}$$

and equalities hold if A has finite left global dimension.

In this theorem, ${}_A\text{Prj}$ and ${}_A\text{Inj}$ denote the classes of projective and injective left A -modules. Theorem E, which is proved in Section 3, generalises well-known descriptions of semi-projective and semi-injective complexes over a ring with finite left global dimension; see Avramov and Foxby [2] and Example 3.3 for further details. By contrast, a complete understanding of ${}^\perp\mathcal{E}$ and \mathcal{E}^\perp is generally not available, not even for chain complexes. An illustration of how complicated the class ${}^\perp\mathcal{E}$ can be is found in Appendix A.

This concludes the introduction. In the next section we recap some main definitions and results from [18] which are key to understanding the present paper.

2. PREREQUISITES

We start by recalling some general facts about (abelian) model categories.

2.1. *Let W be a collection of morphisms in a category \mathcal{C} . By definition, the localization of \mathcal{C} with respect to W is a category $\mathcal{C}[W^{-1}]$ together with a functor $\mathcal{C} \rightarrow \mathcal{C}[W^{-1}]$ satisfying the expected universal property; see for example Krause [27, §1.1] or Weibel [34, §10.3]. There is a standard construction of the category $\mathcal{C}[W^{-1}]$ and the functor $\mathcal{C} \rightarrow \mathcal{C}[W^{-1}]$ that always works if one ignores set-theoretical difficulties. In this construction, the category $\mathcal{C}[W^{-1}]$ has the same objects as \mathcal{C} ; however, in general, the hom “sets” in $\mathcal{C}[W^{-1}]$ need not be small, so the “category” $\mathcal{C}[W^{-1}]$ might not exist within the same set-theoretic universe as \mathcal{C} . In the situation where \mathcal{C} is a model category and W is the collection of weak equivalences*

in \mathcal{C} (which is the situation of interest in this paper), the localization $\mathcal{C}[W^{-1}]$ does exist; it is called the homotopy category of \mathcal{C} and denoted by $\mathrm{Ho}(\mathcal{C})$; see e.g. Hovey [19, §1.2].

We assume that the reader has basic knowledge of *cotorsion pairs*; in particular, (s)he must know what a cotorsion pair is and what it means for such a pair to be *complete* or *hereditary*. The book [14] (Chaps. 2 and 3) by Göbel and Trlifaj is an excellent account on cotorsion pairs in module categories. Cotorsion pairs in general abelian or exact categories are treated in various papers, see e.g. Saorín and Šťovíček [33] and Gillespie [11, 12].

2.2. *Let \mathcal{M} be an abelian category. An abelian model structure on \mathcal{M} is a model structure compatible with the abelian structure in the sense of [20, Dfn. 2.1]. By Thm. 2.2 in loc. cit. any abelian model structure on \mathcal{M} corresponds to a triple $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ of classes of objects in \mathcal{M} for which \mathcal{W} is thick and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{W} \cap \mathcal{R})$ are complete cotorsion pairs. Such a triple is called a Hovey triple in \mathcal{M} ; in fact, \mathcal{Q} is the class of cofibrant objects, \mathcal{R} is the class of fibrant objects, and \mathcal{W} is the class of trivial objects, i.e. objects that are weakly equivalent to 0. If the complete cotorsion pairs $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{W} \cap \mathcal{R})$ are hereditary, then $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ is called a hereditary Hovey triple and the associated abelian model structure on \mathcal{M} is called a hereditary abelian model structure. In this case, the full subcategory $\mathcal{M}_{\mathrm{cf}} = \mathcal{Q} \cap \mathcal{R}$ of cofibrant and fibrant (also called bifibrant) objects in \mathcal{M} is a Frobenius exact category whose class of pro-injective objects is $\mathcal{Q} \cap \mathcal{W} \cap \mathcal{R}$, see Gillespie [12, Prop. 4.2]. The stable category $\mathrm{Stab}(\mathcal{M}_{\mathrm{cf}}) = (\mathcal{Q} \cap \mathcal{R}) / (\mathcal{Q} \cap \mathcal{W} \cap \mathcal{R})$ is by [12, Thm. 4.3] triangulated equivalent to the homotopy category $\mathrm{Ho}(\mathcal{M}) = \mathcal{M}[\{\text{weak equivalences}\}^{-1}]$; in fact, the inclusion functor $\mathcal{M}_{\mathrm{cf}} \rightarrow \mathcal{M}$ induces a triangulated equivalence,*

$$(\#2) \quad \mathrm{Stab}(\mathcal{M}_{\mathrm{cf}}) \xrightarrow{\simeq} \mathrm{Ho}(\mathcal{M})$$

whose quasi-inverse is the bifibrant replacement functor $\mathcal{R}\mathcal{Q}$ described in [12, Lem. 2.5]: For $X \in \mathcal{M}$ write $\mathcal{Q}X$ for a cofibrant replacement and $\mathcal{R}X$ for a fibrant replacement of X . A bifibrant replacement of X is $\mathcal{R}\mathcal{Q}X$. Note that $\mathcal{Q}X \in \mathcal{Q}$, $\mathcal{R}X \in \mathcal{R}$, and $\mathcal{R}\mathcal{Q}X \in \mathcal{Q} \cap \mathcal{R}$.

As explained in Happel [15, Chap. I§2], the stable category of any Frobenius category is triangulated in a canonical way, so the triangulated equivalence (#2) provides a rather explicit description of the triangulated structure on $\mathrm{Ho}(\mathcal{M})$. This will be useful for us.

We now give a brief summary of some definitions and results from [18], which are key to this paper.

2.3 (Blanket assumption). *Unless otherwise stated, \mathbb{k} denotes in this paper any commutative ring, \mathcal{Q} any small \mathbb{k} -preadditive category, and A any \mathbb{k} -algebra.*

2.4 (Notation). *We will use the notation from [18]. In particular, ${}_A\mathrm{Mod}$ denotes the category of left A -modules and ${}_{\mathcal{Q}, A}\mathrm{Mod}$ is the category of \mathbb{k} -linear functors $\mathcal{Q} \rightarrow {}_A\mathrm{Mod}$, see [18, Dfn. 3.1]. Further symbols along with references to where they appear in [18] will follow.*

2.5. Recall from [18, Cor. 3.9] that for every $q \in Q$ there is an adjoint triple (F_q, E_q, G_q) :

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & F_q & \\
 & \curvearrowright & \\
 {}_{Q,A}\text{Mod} & \xrightarrow{E_q} & {}_A\text{Mod} \\
 & \curvearrowleft & \\
 & G_q &
 \end{array} & \text{given by} &
 \begin{array}{l}
 F_q(M) = Q(q, -) \otimes_{\mathbb{k}} M \\
 E_q(X) = X(q) \\
 G_q(M) = \text{Hom}_{\mathbb{k}}(Q(-, q), M)
 \end{array}
 \end{array}$$

for $M \in {}_A\text{Mod}$ and $X \in {}_{Q,A}\text{Mod}$. Note that E_q is the evaluation functor at q .

Some of the definitions below are borrowed from Dell’Ambrogio, Stevenson, and Šťovíček [6, Dfns. 4.1, 4.5 and Rmk. 4.7]. Note that Serre functors were originally introduced by Bondal and Kapranov in [4] (in the case where \mathbb{k} is a field). The definitions play a key role in [18] and in the present work.

2.6. We consider the following conditions on \mathbb{k} and Q (which may or may not hold).

- (1) Hom-finiteness: Each hom \mathbb{k} -module $Q(p, q)$ is finitely generated and projective.
- (2) Local boundedness: For each $q \in Q$ there are only finitely many objects in Q mapping nontrivially into or out of q , that is, the following sets are finite:

$$\begin{aligned}
 N_-(q) &= \{p \in Q \mid Q(p, q) \neq 0\} & \text{and} \\
 N_+(q) &= \{r \in Q \mid Q(q, r) \neq 0\}.
 \end{aligned}$$

- (3) Existence of a Serre functor: There exists a \mathbb{k} -linear autoequivalence $\mathbb{S}: Q \rightarrow Q$ and a natural isomorphism $Q(p, q) \cong \text{Hom}_{\mathbb{k}}(Q(q, \mathbb{S}(p)), \mathbb{k})$.
- (4) Strong Retraction Property: For $q \in Q$ the unit map $\mathbb{k} \rightarrow Q(q, q)$ given by $x \mapsto x \cdot \text{id}_q$ has a \mathbb{k} -module retraction; whence there is a \mathbb{k} -module decomposition:

$$Q(q, q) = (\mathbb{k} \cdot \text{id}_q) \oplus \tau_q.$$

Moreover, the \mathbb{k} -submodules τ_q are compatible with composition in Q as follows:

- (†) $\tau_q \circ \tau_q \subseteq \tau_q$ for all $q \in Q$.
- (‡) $Q(q, p) \circ Q(p, q) \subseteq \tau_p$ for all $p \neq q$ in Q .

2.7. The Strong Retraction Property, i.e. condition (4) in 2.6, makes it possible to define the pseudo-radical τ . This important ideal in Q is explained in detail in [18, Rmk. 7.4 and Lem. 7.7]. Using the pseudo-radical one can for every $q \in Q$ define stalk functors:

$$S\langle q \rangle = Q(q, -)/\tau(q, -) \in {}_Q\text{Mod} \quad \text{and} \quad S\{q\} = Q(-, q)/\tau(-, q) \in \text{Mod}_Q,$$

see [18, Dfn. 7.9 and Lem. 7.10]. Using the stalk functors one can for each $q \in Q$ and $i \geq 0$ define (co)homology functors:

$$\mathbb{H}_{[q]}^i = \text{Ext}_Q^i(S\langle q \rangle, ?) \quad \text{and} \quad \mathbb{H}_i^{[q]} = \text{Tor}_i^Q(S\{q\}, ?),$$

which are functors from ${}_{Q,A}\text{Mod}$ to ${}_A\text{Mod}$; see [18, Dfn. 7.11].

2.8. Assume that Q satisfies the Strong Retraction Property, i.e. condition (4) in 2.6. Recall from [18, Prop. 7.15] (see also Setup 7.13 in loc. cit.) that for every $q \in Q$ there is an adjoint triple (C_q, S_q, K_q) :

$$\begin{array}{ccc}
 & C_q & \\
 \curvearrowright & & \curvearrowleft \\
 {}_A\text{Mod} & \xrightarrow{S_q} & {}_{Q,A}\text{Mod} \\
 \curvearrowleft & & \curvearrowright \\
 & K_q &
 \end{array}
 \quad \text{given by} \quad
 \begin{array}{l}
 C_q(X) = S\langle q \rangle \otimes_Q X \\
 S_q(M) = S\langle q \rangle \otimes_{\mathbb{k}} M \\
 K_q(X) = \text{Hom}_Q(S\langle q \rangle, X)
 \end{array}$$

for $M \in {}_A\text{Mod}$ and $X \in {}_{Q,A}\text{Mod}$.

Many of the results in [18] and in this paper require the following setup.

Setup 2.9. The commutative ring \mathbb{k} and the \mathbb{k} -preadditive category Q from 2.3 satisfy:

- Q is Hom-finite, locally bounded, has a Serre functor, and has the Strong Retraction Property, that is, all four conditions in 2.6 hold.
- The pseudo-radical τ from 2.7 is nilpotent, that is, $\tau^N = 0$ for some $N \in \mathbb{N}$.
- The ring \mathbb{k} is noetherian and hereditary (e.g. \mathbb{k} is a field or $\mathbb{k} = \mathbb{Z}$).
- A can be any \mathbb{k} -algebra.

Let us explain the importance and the power of this setup:

2.10. Assume Setup 2.9. In this case, there is by [18, Dfn. 4.1] an equality:

$$\begin{aligned}
 \mathcal{E} &= \{X \in {}_{Q,A}\text{Mod} \mid X^\natural \text{ has finite projective dimension in } {}_Q\text{Mod}\} \\
 &= \{X \in {}_{Q,A}\text{Mod} \mid X^\natural \text{ has finite injective dimension in } {}_Q\text{Mod}\},
 \end{aligned}$$

where $(-)^{\natural}$ is the forgetful functor, see [18, Dfn. 3.2]. Objects in \mathcal{E} are called exact and the class \mathcal{E} is by [18, Thm. 6.1] part of two different hereditary Hovey triples,

$$(\mathcal{Q}, \mathcal{W}, \mathcal{R}) = ({}^\perp \mathcal{E}, \mathcal{E}, {}_{Q,A}\text{Mod}) \quad \text{and} \quad (\mathcal{Q}, \mathcal{W}, \mathcal{R}) = ({}_{Q,A}\text{Mod}, \mathcal{E}, \mathcal{E}^\perp),$$

which, as explained in 2.2, yield two different model structures on ${}_{Q,A}\text{Mod}$:

- The projective model structure on ${}_{Q,A}\text{Mod}$ where ${}^\perp \mathcal{E}$ is the class of cofibrant objects, \mathcal{E} is the class of trivial objects, and every object is fibrant.
- The injective model structure on ${}_{Q,A}\text{Mod}$ where \mathcal{E}^\perp is the class of fibrant objects, \mathcal{E} is the class of trivial objects, and every object is cofibrant.

These two model structures have by [18, Prop. 6.3] the same weak equivalences and the associated homotopy category is, as in [18, Dfn. 6.4], denoted by

$$\mathcal{D}_Q(A) = \text{Ho}({}_{Q,A}\text{Mod}).$$

It is called the Q -shaped derived category of A . By the general theory from 2.2, the category ${}^\perp \mathcal{E}$, respectively, \mathcal{E}^\perp , is a Frobenius exact category whose class of pro-injective objects is ${}^\perp \mathcal{E} \cap \mathcal{E} = {}_{Q,A}\text{Prj}$ (the projective objects in ${}_{Q,A}\text{Mod}$), respectively, $\mathcal{E} \cap \mathcal{E}^\perp = {}_{Q,A}\text{Inj}$ (the injective objects in ${}_{Q,A}\text{Mod}$). As mentioned in 2.2 and recorded in [18, Thm. 6.5] there are equivalences of triangulated categories,

$$\frac{{}^\perp \mathcal{E}}{{}_{Q,A}\text{Prj}} \simeq \mathcal{D}_Q(A) \simeq \frac{\mathcal{E}^\perp}{{}_{Q,A}\text{Inj}}.$$

Finally, the (co)homology functors $\mathbb{H}_{[q]}^i, \mathbb{H}_i^{[q]}: {}_{Q,A}\text{Mod} \rightarrow {}_A\text{Mod}$ from 2.7 detect exact objects and weak equivalences in ${}_{Q,A}\text{Mod}$ as described in [18, Thms. 7.1

and 7.2]. In particular, the functors $\mathbb{H}_{[q]}^i$ and $\mathbb{H}_i^{[q]}$ map weak equivalences into isomorphisms, and hence they induce well-defined functors on the Q -shaped derived category $\mathcal{D}_Q(A)$.

Not all results in this paper require the full strength of Setup 2.9; as the reader will notice we sometimes only assume some of the conditions in 2.6. This is the case e.g. Lemma 3.4 and Proposition 3.7, and it is also the case for a number of results in Sections 4 and 5.

3. (CO)FIBRANT OBJECTS IN ${}_{Q,A}\text{Mod}$

In general, the class ${}^\perp\mathcal{E}$ of cofibrant objects in the projective model structure and the class \mathcal{E}^\perp of fibrant objects in the injective model structure on ${}_{Q,A}\text{Mod}$ (see 2.10) are hard to perceive. The main purpose of this section is to prove Theorem E from Section 1, which gives hands-on descriptions of the classes ${}^\perp\mathcal{E}$ and \mathcal{E}^\perp in the case where A has finite global dimension. Theorem E is also an important ingredient in the proof of Theorem B (from Section 1), which is established in Section 5. Another goal in this section is Theorem 3.9, which shows how that classes ${}^\perp\mathcal{E}$ and \mathcal{E}^\perp behave in short exact sequences.

To parse the next definition, recall from [16, 6.4 and 6.7] the definitions of *filtrations* and *cofiltrations* with respect to some class of objects in an abelian category.

Definition 3.1. For a class \mathcal{C} of objects in a bicomplete abelian category \mathcal{M} we set

$$\begin{aligned} \text{Filt}(\mathcal{C}) &= \{M \in \mathcal{M} \mid M \text{ has a } \mathcal{C}\text{-filtration}\}, \\ \text{coFilt}(\mathcal{C}) &= \{M \in \mathcal{M} \mid M \text{ has a } \mathcal{C}\text{-cofiltration}\}. \end{aligned}$$

This definition is important in the next result, and so are the (stalk) functors S_q from 2.8. Furthermore, ${}_A\text{Prj}$ and ${}_A\text{Inj}$ denote the classes of projective and injective left A -modules.

Proposition 3.2. *Assume Setup 2.9. The following assertions hold.*

- (a) *With $\mathcal{P} = \{S_q(P) \mid q \in Q \text{ and } P \in {}_A\text{Prj}\}$ there are inclusions,*

$$\text{Filt}(\mathcal{P}) \subseteq {}^\perp\mathcal{E} \subseteq \{X \in {}_{Q,A}\text{Mod} \mid X(q) \in {}_A\text{Prj} \text{ for all } q \in Q\}.$$
- (b) *With $\mathcal{I} = \{S_q(I) \mid q \in Q \text{ and } I \in {}_A\text{Inj}\}$ there are inclusions,*

$$\text{coFilt}(\mathcal{I}) \subseteq \mathcal{E}^\perp \subseteq \{X \in {}_{Q,A}\text{Mod} \mid X(q) \in {}_A\text{Inj} \text{ for all } q \in Q\}.$$

Proof. (a): To prove the first inclusion, it suffices by Eklof’s lemma [16, Lem. 6.6] to show that \mathcal{P} is a subset of ${}^\perp\mathcal{E}$. Let $q \in Q$ and $P \in {}_A\text{Prj}$ be given. Recall from 2.8 that the functor S_q is exact with right adjoint K_q . The first right derived functor R^1K_q is the cohomology functor $\mathbb{H}_{[q]}^1$ from 2.7. Thus, for every $E \in \mathcal{E}$ one has $R^1K_q(E) = \mathbb{H}_{[q]}^1(E) = 0$ by [18, Thm. 7.1], and the first isomorphism below now follows from [17, Lem. 1.3]. The second isomorphism holds as $P \in {}_A\text{Prj}$.

$$\text{Ext}_{Q,A}^1(S_q(P), E) \cong \text{Ext}_A^1(P, K_q(E)) \cong 0.$$

This proves that $S_q(P)$ belongs to ${}^\perp\mathcal{E}$, and hence $\mathcal{P} \subseteq {}^\perp\mathcal{E}$, as desired.

To prove the second inclusion, let $X \in {}^\perp\mathcal{E}$ and $q \in Q$ be given. The evaluation functor E_q at q has by 2.5 a right adjoint G_q , so there is a natural isomorphism,

$$\text{Hom}_A(X(q), -) = \text{Hom}_A(E_q(X), -) \cong \text{Hom}_{Q,A}(X, G_q(-)).$$

Thus, we must prove that the last functor above is exact. Let $\xi = 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence in ${}_A\text{Mod}$. As the category Q is assumed to be Hom-finite, the functor G_q is exact by [18, Cor. 3.9(b)] and hence $G_q(\xi)$ is an exact sequence in ${}_{Q,A}\text{Mod}$. To prove exactness of the sequence $\text{Hom}_{Q,A}(X, G_q(\xi))$, it suffices to argue that one has

$$(\#3) \quad \text{Ext}_{Q,A}^1(X, G_q(M')) = 0.$$

By the assumptions on Q , we can apply [18, Thm. 7.1 and Lem. 7.14(b)] to conclude that $G_q(M') \in \mathcal{E}$; and since one has $X \in {}^\perp \mathcal{E}$, it follows that $(\#3)$ holds.

(b): Dual to the proof of part (a). Note that instead of Eklof’s lemma [16, Lem. 6.6] and [17, Lem. 1.3] one uses Trlifaj’s lemma [16, Lem. 6.8] and [17, Lem. 1.2]. □

We are now in a position to prove Theorem E from Section 1.

Proof of Theorem E. The inclusions were established in Proposition 3.2. Now assume that A has finite left global dimension. We only prove that the first inclusion is an equality; a similar argument shows that the second inclusion is an equality. Let $X \in {}_{Q,A}\text{Mod}$ be an object that satisfies $X(q) \in {}_A\text{Prj}$ for all $q \in Q$. We must argue that X belongs to ${}^\perp \mathcal{E}$. Since $({}^\perp \mathcal{E}, \mathcal{E})$ is a complete cotorsion pair in ${}_{Q,A}\text{Mod}$, see 2.10 (or [18, Thm. 5.5]), there exists an exact sequence,

$$(\#4) \quad 0 \longrightarrow E \longrightarrow P \longrightarrow X \longrightarrow 0$$

with $P \in {}^\perp \mathcal{E}$ and $E \in \mathcal{E}$. The goal is to show that this sequence splits; this will imply that $X \in {}^\perp \mathcal{E}$ since $P \in {}^\perp \mathcal{E}$. We have $P(q), X(q) \in {}_A\text{Prj}$ for all $q \in Q$ by Proposition 3.2(a) and by assumption. As the sequence $0 \rightarrow E(q) \rightarrow P(q) \rightarrow X(q) \rightarrow 0$ is exact, it follows that also $E(q) \in {}_A\text{Prj}$ for all $q \in Q$.

At this point, we pause to make a general observation: Let $n \in \mathbb{N}_0$ be the left global dimension of A . Evidently, the category ${}_A\text{Mod}$ is locally n -Gorenstein in the sense of [6, Dfns. 2.1 and 2.5], so by definition (see the remarks above [6, Thm. 4.6]), this means that A is an n -Gorenstein algebra. Hence [6, Cor. 4.8]¹ implies that an object $M \in {}_{Q,A}\text{Mod}$ is Gorenstein projective if and only if the A -module $M(q)$ is Gorenstein projective for every $q \in Q$. Since A has finite left global dimension, the latter condition just means that $M(q) \in {}_A\text{Prj}$ for every $q \in Q$, see [8, Prop. 10.2.3].

By the observation above, it follows that X, P , and E are Gorenstein projective objects in ${}_{Q,A}\text{Mod}$. We will now show that E is even a projective object in ${}_{Q,A}\text{Mod}$. Once this has been shown, it follows immediately from the definition of Gorenstein projective objects, see [7, Dfn. 2.20], that $\text{Ext}_{Q,A}^1(X, E) = 0$, and thus $(\#4)$ will split. We already know that $E \in \mathcal{E}$. Thus, to show that E is projective, we need by [18, Thms. 7.1 and 7.29] to argue that the A -module $C_q(E)$ is projective for every $q \in Q$, where C_q is the functor from 2.8. As E is a Gorenstein projective object in ${}_{Q,A}\text{Mod}$ there is, by definition, an exact sequence,

$$(\#5) \quad 0 \longrightarrow E \longrightarrow P^0 \longrightarrow \dots \longrightarrow P^{n-1} \longrightarrow E' \longrightarrow 0$$

in ${}_{Q,A}\text{Mod}$ where each P^i is projective (and E' is Gorenstein projective, but that is not important here). By another application of [18, Thms. 7.1 and 7.29], each

¹Note that [6, Cor. 4.8] deals with $A \otimes Q$ -modules M , i.e. \mathbb{k} -linear functors $M: A \otimes Q \rightarrow \mathbb{k}\text{Mod}$, where $A \otimes Q$ is the *extension* of Q by A defined above [6, Thm. 4.6]. However, it is easy to see that the category ${}_{A \otimes Q}\text{Mod}$ of $A \otimes Q$ -modules is equivalent to the category ${}_{Q,A}\text{Mod}$ of \mathbb{k} -linear functors $Q \rightarrow {}_A\text{Mod}$.

P^i belongs to \mathcal{E} , and hence E' belongs to \mathcal{E} by the last assertion in [18, Thm. 4.4]. Now fix $q \in Q$. The right exact functor C_q is, in general, not exact; indeed, its i th left derived functor $L_i C_q$ is the homology functor $\mathbb{H}_i^{[q]}$ from 2.7. Since E' is in \mathcal{E} we have $L_i C_q(E') = \mathbb{H}_i^{[q]}(E') = 0$ for all $i > 0$ by [18, Thm. 7.1], and thus (#5) does, in fact, induce an exact sequence,

$$0 \longrightarrow C_q(E) \longrightarrow C_q(P^0) \longrightarrow \cdots \longrightarrow C_q(P^{n-1}) \longrightarrow C_q(E') \longrightarrow 0,$$

of left A -modules. Each module $C_q(P^i)$ is projective by [18, Thm. 7.29], and the module $C_q(E')$ has projective dimension at most n ($=$ the left global dimension of A). The exact sequence therefore shows that $C_q(E)$ is projective, as desired. \square

Example 3.3. Consider the situation where Q is the mesh category of the repetitive quiver of $\vec{A}_2 = \bullet \rightarrow \bullet$; in other words, Q is the path category of the quiver (#1) with the relations $\partial^2 = 0$. In this case, ${}_{Q,A}\text{Mod}$ is equivalent to the category $\text{Ch } A$ of (chain) complexes of left A -modules, see [18, Rmk. 8.5 and Exa. 8.13]. For the class \mathcal{E} one has:

$$\begin{aligned} \mathcal{E} &= \{\text{exact (or acyclic) complexes}\}, \\ {}^\perp \mathcal{E} &= \{\text{semi-projective (or DG-projective) complexes}\}, \quad \text{and} \\ \mathcal{E}^\perp &= \{\text{semi-injective (or DG-injective) complexes}\}. \end{aligned}$$

Indeed, the first equality follows easily from [18, Thm. 7.1] and the other two follow from García Rozas [10, Props. 2.3.4 and 2.3.5].

In this case, the class \mathcal{P} from Proposition 3.2 is just $\{\Sigma^q P \mid q \in \mathbb{Z} \text{ and } P \in {}_A\text{Prj}\}$, where Σ is the shift functor on $\text{Ch } A$. It is easily seen that every bounded below complex of projective A -modules, $P_\bullet = \cdots \rightarrow P_{u+2} \rightarrow P_{u+1} \rightarrow P_u \rightarrow 0 \rightarrow 0 \rightarrow \cdots$, is in $\text{Filt}(\mathcal{P})$. Hence every such complex is semi-projective by the first inclusion in part (a) of Proposition 3.2. Similarly, the first inclusion in part (b) of the same result shows that every bounded above complex of injective A -modules, $I_\bullet = \cdots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow I_v \rightarrow I_{v-1} \rightarrow I_{v-2} \rightarrow \cdots$, is semi-injective. This is of course well-known. The second inclusions in parts (a) and (b) of Proposition 3.2 assert that every semi-projective/injective complex must consist of projective/injective modules, which is also well-known. Theorem E in Section 1 shows that over a ring with finite global dimension, a semi-projective/injective complex is *nothing but* a complex consisting of projective/injective modules. This special case of Theorem E can be found in [2, Prop. 3.4] by Avramov and Foxby. Applications of their result (and related results) can be found in [21, 22] by Iacob and Iyengar.

An important fact, which we shall now prove, is that when Q is Hom-finite and has a Serre functor, the family $\{F_q\}_{q \in Q}$ is just a permutation of the family $\{G_q\}_{q \in Q}$, see 2.5.

Lemma 3.4. *Assume that Q is Hom-finite and has a Serre functor, i.e. conditions (1) and (3) in 2.6 hold. For every $q \in Q$ there is a natural isomorphism of functors $G_{S(q)} \cong F_q$.*

Proof. Let $q \in Q$ be given. For every $M \in {}_A\text{Mod}$ and $p \in Q$ one has the following isomorphisms of left A -modules:

$$\begin{aligned} G_{\mathbb{S}(q)}(M)(p) &\cong \text{Hom}_{\mathbb{k}}(Q(p, \mathbb{S}(q)), M) \\ &\cong \text{Hom}_{\mathbb{k}}(Q(p, \mathbb{S}(q)), \mathbb{k}) \otimes_{\mathbb{k}} M \\ &\cong Q(q, p) \otimes_{\mathbb{k}} M \\ &\cong F_q(M)(p). \end{aligned}$$

In this computation, the first and last isomorphisms hold by the definitions of the functors $G_{\mathbb{S}(q)}$ and F_q , see 2.5. The second isomorphism is induced by the so-called tensor evaluation map, which in this case is an isomorphism as the \mathbb{k} -module $Q(p, \mathbb{S}(q))$ is finitely generated and projective by Hom-finiteness of Q . This isomorphism can easily be proved but it can also be found in e.g. [23, Lem. 1.1]. The third isomorphism holds by the definition of the Serre functor. All the isomorphisms above are natural in p and M . □

Definition 3.5. A family $\{X_i\}_{i \in I}$ of objects in ${}_{Q,A}\text{Mod}$ is called *locally finite* if for each $q \in Q$ the set $\{i \in I \mid X_i(q) \neq 0\}$ is finite.

Lemma 3.6. *If $\{X_i\}_{i \in I}$ is a locally finite family of objects in ${}_{Q,A}\text{Mod}$, then there is an isomorphism $\bigoplus_{i \in I} X_i \cong \prod_{i \in I} X_i$.*

Proof. Straightforward. □

Proposition 3.7. *Assume that Q is locally bounded, that is, condition (2) in 2.6 holds. For every family $\{M_q\}_{q \in Q}$ of objects in ${}_A\text{Mod}$, there are isomorphisms:*

$$\bigoplus_{q \in Q} F_q(M_q) \cong \prod_{q \in Q} F_q(M_q) \quad \text{and} \quad \bigoplus_{q \in Q} G_q(M_q) \cong \prod_{q \in Q} G_q(M_q).$$

Proof. By Lemma 3.6 it suffices to argue that the families $\{F_q(M_q)\}_{q \in Q}$ and $\{G_q(M_q)\}_{q \in Q}$ are locally finite. We only consider the first family. It must be proved that for every $r \in Q$ the set $\{q \in Q \mid F_q(M_q)(r) \neq 0\}$ is finite. By definition one has $F_q(M_q)(r) = Q(q, r) \otimes_{\mathbb{k}} M_q$, see 2.5, so the desired conclusion follows from the local boundedness of Q , which yields that the set $N_-(r) = \{q \in Q \mid Q(q, r) \neq 0\}$ is finite. □

Definition 3.8. A short exact sequence $\xi = 0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ in ${}_{Q,A}\text{Mod}$ is said to be *objectwise split* if the sequence $E_q(\xi) = \xi(q) = 0 \rightarrow X'(q) \rightarrow X(q) \rightarrow X''(q) \rightarrow 0$ splits in ${}_A\text{Mod}$ for every object $q \in Q$.

We end by proving Theorem 3.9 announced in the beginning of this section. This result plays an important role in the proof of Proposition 5.11, which yields certain canonical conflations in the exact categories ${}^\perp \mathcal{E}$ and \mathcal{E}^\perp .

Theorem 3.9. *Assume that Setup 2.9 holds and let $\xi = 0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ be a short exact sequence in ${}_{Q,A}\text{Mod}$. For the class ${}^\perp \mathcal{E}$ the following assertions hold:*

- (a) *Assume that $X'' \in {}^\perp \mathcal{E}$. One has $X' \in {}^\perp \mathcal{E}$ if and only if $X \in {}^\perp \mathcal{E}$.*
- (b) *Assume that ξ is objectwise split. If one has $X', X \in {}^\perp \mathcal{E}$, then $X'' \in {}^\perp \mathcal{E}$.*

Dually, for the class \mathcal{E}^\perp the following assertions hold:

- (c) *Assume that $X' \in \mathcal{E}^\perp$. One has $X \in \mathcal{E}^\perp$ if and only if $X'' \in \mathcal{E}^\perp$.*
- (d) *Assume that ξ is objectwise split. If one has $X, X'' \in \mathcal{E}^\perp$, then $X' \in \mathcal{E}^\perp$.*

Proof. Parts (a) and (c) follow immediately from the fact that the cotorsion pairs $({}^\perp \mathcal{E}, \mathcal{E})$ and $(\mathcal{E}, \mathcal{E}^\perp)$ are hereditary, see 2.10 (or [18, Thm. 4.4]). It remains to show parts (b) and (d); we only prove (b) as the proof of (d) is dual. Write α for the monomorphism $X' \rightarrow X$ in the given exact sequence ξ . By assumption, $\alpha(q)$ is split monic in ${}_A \text{Mod}$ for every $q \in Q$ and the objects X' and X belong to ${}^\perp \mathcal{E}$. It must be shown that $\text{Ext}_{Q,A}^1(X'', E) = 0$ holds for every $E \in \mathcal{E}$. Fix $E \in \mathcal{E}$. Application of the functor $\text{Hom}_{Q,A}(-, E)$ to the sequence ξ yields an exact sequence,

$$\begin{aligned} \text{Hom}_{Q,A}(X, E) &\xrightarrow{\text{Hom}_{Q,A}(\alpha, E)} \text{Hom}_{Q,A}(X', E) \\ &\longrightarrow \text{Ext}_{Q,A}^1(X'', E) \longrightarrow \text{Ext}_{Q,A}^1(X, E) = 0, \end{aligned}$$

so it suffices to argue that the homomorphism $\text{Hom}_{Q,A}(\alpha, E)$ is surjective. The category ${}_Q, {}_A \text{Mod}$ has enough projectives, see e.g. [18, Prop. 3.12], so there exists an exact sequence $0 \rightarrow E' \rightarrow P \xrightarrow{\pi} E \rightarrow 0$ with P projective. As $P \in {}_Q, {}_A \text{Prj} \subseteq \mathcal{E}$ it follows from [18, Thm. 4.4] that $E' \in \mathcal{E}$. Application of $\text{Hom}_{Q,A}(X', -)$ to this sequence yields an exact sequence,

$$\text{Hom}_{Q,A}(X', P) \xrightarrow{\text{Hom}_{Q,A}(X', \pi)} \text{Hom}_{Q,A}(X', E) \longrightarrow \text{Ext}_{Q,A}^1(X', E') = 0,$$

so the map $\text{Hom}_{Q,A}(X', \pi)$ is surjective. In view of this and of the commutative diagram

$$\begin{array}{ccc} \text{Hom}_{Q,A}(X, P) & \xrightarrow{\text{Hom}_{Q,A}(X, \pi)} & \text{Hom}_{Q,A}(X, E) \\ \text{Hom}_{Q,A}(\alpha, P) \downarrow & & \downarrow \text{Hom}_{Q,A}(\alpha, E) \\ \text{Hom}_{Q,A}(X', P) & \xrightarrow{\text{Hom}_{Q,A}(X', \pi)} & \text{Hom}_{Q,A}(X', E), \end{array}$$

surjectivity of $\text{Hom}_{Q,A}(\alpha, E)$ will follow if we can prove surjectivity of $\text{Hom}_{Q,A}(\alpha, P)$. Recall from [18, Prop. 3.12(a)] that $\{F_q(A)\}_{q \in Q}$ is a family of projective generators of ${}_Q, {}_A \text{Mod}$, and hence the projective object P is a direct summand in an object of the form $\bigoplus_{q \in Q} F_q(A)^{I_q}$ for suitable index sets I_q . Note that one has

$$\bigoplus_{q \in Q} F_q(A)^{I_q} \cong \bigoplus_{q \in Q} F_q(A^{I_q}) \cong \prod_{q \in Q} G_{\mathbb{S}(q)}(A^{I_q}).$$

The first isomorphism holds as each functor F_q preserves coproducts; indeed, F_q is a left adjoint by 2.5. The second isomorphism follows from Lemma 3.4 and Proposition 3.7. These considerations, and the fact that the functor $\text{Hom}_{Q,A}(\alpha, -)$ preserves products, show that without loss of generality we may assume that P has the form $P = G_p(A^{I})$ for some object $p \in Q$ and index set I . Set $M = A^{I}$ and note that one has

$$\text{Hom}_{Q,A}(\alpha, P) = \text{Hom}_{Q,A}(\alpha, G_p(M)) \cong \text{Hom}_A(\alpha(p), M)$$

since G_p is right adjoint of the evaluation functor at p , see 2.5. By assumption, $\alpha(p)$ is split monic, and hence $\text{Hom}_A(\alpha(p), M)$ is (split) surjective, as desired. \square

Remark 3.10. Easy examples show that the conclusions in parts (b) and (d) in Theorem 3.9 fail if the exact sequence ξ is not objectwise split.

4. COMPACT OBJECTS IN $\mathcal{D}_{\mathcal{Q}}(A)$

The purpose of this section is to prove Theorem D from Section 1. We will obtain this theorem as a special case of Theorem 4.5, which is akin to Hovey’s [19, Cor. 7.4.4].

We begin with a familiar looking Ext-formula for the hom sets in the homotopy category of an abelian model category. This formula can certainly be found in various special cases in the literature. Even in the generality below, the formula is probably known to the experts; however, since we were not able to find a reference, we have included a proof. To parse the result, recall from 2.2 that \mathcal{R} and \mathcal{Q} denote the fibrant and cofibrant replacement functors.

Proposition 4.1. *Let \mathcal{M} be an abelian category equipped with a hereditary abelian model structure. Consider the associated triangulated homotopy category $\text{Ho}(\mathcal{M})$ whose translation functor we denote by Σ . For every $X, Y \in \mathcal{M}$ and $n > 0$ there are isomorphisms:*

$$\text{Hom}_{\text{Ho}(\mathcal{M})}(\Sigma^{-n}X, Y) \cong \text{Ext}_{\mathcal{M}}^n(\mathcal{R}\mathcal{Q}X, \mathcal{R}\mathcal{Q}Y) \cong \text{Hom}_{\text{Ho}(\mathcal{M})}(X, \Sigma^n Y).$$

Proof. Since Σ is an autoequivalence on $\text{Ho}(\mathcal{M})$, the abelian groups $\text{Hom}_{\text{Ho}(\mathcal{M})}(\Sigma^{-n}X, Y)$ and $\text{Hom}_{\text{Ho}(\mathcal{M})}(X, \Sigma^n Y)$ are isomorphic, so we only need to prove the second of the asserted isomorphisms. Let $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ be the hereditary Hovey triple that corresponds to the given hereditary abelian model structure on \mathcal{M} via [20, Thm. 2.2]. As explained in 2.2, the bifibrant replacement functor $\mathcal{R}\mathcal{Q}$ yields a triangulated equivalence $\text{Ho}(\mathcal{M}) \rightarrow \text{Stab}(\mathcal{M}_{\text{cf}})$. We will use the standard notation $\underline{\text{Hom}}_{\mathcal{M}}$ for the hom-sets in the stable category, that is, for bifibrant objects A and B in \mathcal{M} one has

$$\text{\#6} \quad \underline{\text{Hom}}_{\mathcal{M}}(A, B) = \text{Hom}_{\mathcal{M}}(A, B) / \mathcal{I}(A, B),$$

where $\mathcal{I}(A, B)$ is the subgroup of $\text{Hom}_{\mathcal{M}}(A, B)$ consisting of morphisms $A \rightarrow B$ in \mathcal{M} that factor through some object in $\mathcal{Q} \cap \mathcal{W} \cap \mathcal{R}$. As $\mathcal{R}\mathcal{Q} : \text{Ho}(\mathcal{M}) \rightarrow \text{Stab}(\mathcal{M}_{\text{cf}})$ is a triangulated equivalence there is an isomorphism,

$$\text{Hom}_{\text{Ho}(\mathcal{M})}(X, \Sigma^n Y) \cong \underline{\text{Hom}}_{\mathcal{M}}(\mathcal{R}\mathcal{Q}X, \Sigma^n \mathcal{R}\mathcal{Q}Y),$$

for every $X, Y \in \mathcal{M}$ and $n \in \mathbb{Z}$. In view of this isomorphism, the second isomorphism in the proposition will follow if we can establish an isomorphism

$$\text{\#7} \quad \underline{\text{Hom}}_{\mathcal{M}}(A, \Sigma^n B) \cong \text{Ext}_{\mathcal{M}}^n(A, B)$$

for all objects $A, B \in \mathcal{Q} \cap \mathcal{R}$ (for example $A = \mathcal{R}\mathcal{Q}X$ and $B = \mathcal{R}\mathcal{Q}Y$) and $n > 0$.

To establish the isomorphism (\#7), we start with the case $n = 1$. By definition of the translation functor Σ in the stable category of a Frobenius category, see Happel [15, Chap. I§2], the (translated) object $\Sigma B \in \mathcal{Q} \cap \mathcal{R}$ fits into a short exact sequence in \mathcal{M} ,

$$\text{\#8} \quad 0 \longrightarrow B \longrightarrow W \xrightarrow{\theta} \Sigma B \longrightarrow 0,$$

where W is in $\mathcal{Q} \cap \mathcal{W} \cap \mathcal{R}$. Application of the functor $\text{Hom}_{\mathcal{M}}(A, -)$ to this sequence yields the exact sequence

$$\text{Hom}_{\mathcal{M}}(A, W) \xrightarrow{\text{Hom}_{\mathcal{M}}(A, \theta)} \text{Hom}_{\mathcal{M}}(A, \Sigma B) \longrightarrow \text{Ext}_{\mathcal{M}}^1(A, B) \longrightarrow \text{Ext}_{\mathcal{M}}^1(A, W) = 0;$$

here $\text{Ext}_{\mathcal{M}}^1(A, W)$ is zero as $(\mathcal{Q}, \mathcal{W} \cap \mathcal{R})$ is a cotorsion pair, A is in $\mathcal{Q} \cap \mathcal{R} \subseteq \mathcal{Q}$, and W is in $\mathcal{Q} \cap \mathcal{W} \cap \mathcal{R} \subseteq \mathcal{W} \cap \mathcal{R}$. Thus, $\text{Ext}_{\mathcal{M}}^1(A, B)$ is the cokernel of $\text{Hom}_{\mathcal{M}}(A, \theta)$ and the isomorphism (#7), still for $n = 1$, will follow if we can show the equality

$$\text{Im Hom}_{\mathcal{M}}(A, \theta) = \mathcal{I}(A, \Sigma B).$$

The inclusion “ \subseteq ” is clear as W belongs to $\mathcal{Q} \cap \mathcal{W} \cap \mathcal{R}$. Conversely, let $\alpha: A \rightarrow \Sigma B$ be any morphism that factors through an object in $\mathcal{Q} \cap \mathcal{W} \cap \mathcal{R}$, i.e. there is some factorization,

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & \Sigma B \\ \text{\scriptsize } \varphi \swarrow & & \searrow \text{\scriptsize } \psi \\ & V & \end{array}$$

with $V \in \mathcal{Q} \cap \mathcal{W} \cap \mathcal{R}$. Application of the functor to $\text{Hom}_{\mathcal{M}}(V, -)$ to the short exact sequence (#8) yields an exact sequence

$$\text{Hom}_{\mathcal{M}}(V, W) \xrightarrow{\text{Hom}_{\mathcal{M}}(V, \theta)} \text{Hom}_{\mathcal{M}}(V, \Sigma B) \longrightarrow \text{Ext}_{\mathcal{M}}^1(V, B) = 0 ;$$

here $\text{Ext}_{\mathcal{M}}^1(V, B)$ is zero since $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ is a cotorsion pair, V is in $\mathcal{Q} \cap \mathcal{W} \cap \mathcal{R} \subseteq \mathcal{Q} \cap \mathcal{W}$, and B is in $\mathcal{Q} \cap \mathcal{R} \subseteq \mathcal{R}$. Thus the homomorphism $\text{Hom}_{\mathcal{M}}(V, \theta)$ is surjective, so there exists a morphism $\omega: V \rightarrow W$ in \mathcal{M} with $\theta\omega = \psi$. Therefore $\alpha = \psi\varphi = \theta\omega\varphi = \text{Hom}_{\mathcal{M}}(A, \theta)(\omega\varphi)$. This proves the other inclusion “ \supseteq ” and hence the isomorphism in (#7) is proved for $n = 1$.

Now assume, by induction, that for some $n > 0$ an isomorphism (#7) exists for every $A, B \in \mathcal{Q} \cap \mathcal{R}$. As also ΣB belongs to $\mathcal{Q} \cap \mathcal{R}$, the induction hypothesis yield:

$$\text{\#9) } \underline{\text{Hom}}_{\mathcal{M}}(A, \Sigma^{n+1}B) \cong \underline{\text{Hom}}_{\mathcal{M}}(A, \Sigma^n \Sigma B) \cong \text{Ext}_{\mathcal{M}}^n(A, \Sigma B).$$

Moreover, by applying $\text{Hom}_{\mathcal{M}}(A, -)$ to the short sequence (#8) we get an exact sequence

$$0 = \text{Ext}_{\mathcal{M}}^n(A, W) \longrightarrow \text{Ext}_{\mathcal{M}}^n(A, \Sigma B) \longrightarrow \text{Ext}_{\mathcal{M}}^{n+1}(A, B) \longrightarrow \text{Ext}_{\mathcal{M}}^{n+1}(A, W) = 0 ;$$

note that $\text{Ext}_{\mathcal{M}}^i(A, W) = 0$ for all $i \geq 1$ as $(\mathcal{Q}, \mathcal{W} \cap \mathcal{R})$ is a hereditary cotorsion pair, A is in $\mathcal{Q} \cap \mathcal{R} \subseteq \mathcal{Q}$, and W is in $\mathcal{Q} \cap \mathcal{W} \cap \mathcal{R} \subseteq \mathcal{W} \cap \mathcal{R}$. Hence $\text{Ext}_{\mathcal{M}}^n(A, \Sigma B) \cong \text{Ext}_{\mathcal{M}}^{n+1}(A, B)$, which in combination with (#9) yields an isomorphism $\underline{\text{Hom}}_{\mathcal{M}}(A, \Sigma^{n+1}B) \cong \text{Ext}_{\mathcal{M}}^{n+1}(A, B)$. □

Next we recall some standard definitions.

Definition 4.2. Let \mathcal{A} be an additive category (e.g. an abelian or a triangulated category) with set-indexed coproducts. An object $C \in \mathcal{A}$ is called *compact* if the functor $\text{Hom}_{\mathcal{A}}(C, -)$ preserves coproducts, that is, if the canonical map

$$\bigoplus_{i \in I} \text{Hom}_{\mathcal{A}}(C, X_i) \longrightarrow \text{Hom}_{\mathcal{A}}(C, \bigoplus_{i \in I} X_i)$$

is an isomorphism for every (set-indexed) family $\{X_i\}_{i \in I}$ of objects in \mathcal{A} . We set

$$\mathcal{A}^c = \{C \in \mathcal{A} \mid C \text{ is compact}\}.$$

Definition 4.3. Let \mathcal{T} be a triangulated category with set-indexed coproducts.

- One says that \mathcal{T} is *compactly generated* if there exists a set (as opposed to a proper class) \mathbb{S} of compact objects in \mathcal{T} with the property that the only object $X \in \mathcal{T}$ that satisfies $\text{Hom}_{\mathcal{T}}(S, X) = 0$ for every $S \in \mathbb{S}$ is $X = 0$. See Neeman [29, Dfn. 1.7].

Next, let \mathcal{S} be a triangulated subcategory of \mathcal{T} .

- One says that \mathcal{S} is *thick* (or *épaisse*) if it is closed under direct summands in \mathcal{T} . See Neeman [30, Dfn. 2.1.6] (and Rickard [31, Prop. 1.3]).
- One says that \mathcal{S} is *localizing* if it is closed under coproducts in \mathcal{T} . Such a subcategory is thick, see Bökstedt and Neeman [3, Dfn. 1.3, Rmk. 1.4, and Prop. 3.2].

Finally, let \mathbb{S} be any collection (usually a set) of objects in \mathcal{T} .

- Write $\text{Thick}(\mathbb{S})$ for the *thick subcategory generated by* \mathbb{S} , that is, the smallest thick triangulated subcategory of \mathcal{T} containing \mathbb{S} .
- Write $\text{Loc}(\mathbb{S})$ for the *localizing subcategory generated by* \mathbb{S} , that is, the smallest localizing triangulated subcategory of \mathcal{T} containing \mathbb{S} .

The next two results, Lemma 4.4 and Theorem 4.5, are akin to Hovey’s [19, Thm. 7.4.3 and Cor. 7.4.4]. Our proof of Theorem 4.5 uses the Ext-formula from Proposition 4.1.

Lemma 4.4. *Let \mathcal{M} be a cocomplete abelian category equipped with a hereditary abelian model structure, and assume that the class \mathcal{R} of fibrant objects is closed under coproducts in \mathcal{M} (for example, if $\mathcal{R} = \mathcal{M}$). Let C be an object in \mathcal{M} that is both cofibrant and fibrant. If C is compact in \mathcal{M} , then C is also compact in the homotopy category $\text{Ho}(\mathcal{M})$.*

Proof. Let $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ be the hereditary Hovey triple that corresponds to the given hereditary abelian model structure on \mathcal{M} . As the object $C \in \mathcal{M}$ is both cofibrant and fibrant, it can be viewed as an object in the stable category $\text{Stab}(\mathcal{M}_{\text{cf}}) = (\mathcal{Q} \cap \mathcal{R}) / (\mathcal{Q} \cap \mathcal{W} \cap \mathcal{R})$. As this category is equivalent to $\text{Ho}(\mathcal{M})$, it suffices to show that compactness of C in \mathcal{M} implies compactness of C in $\text{Stab}(\mathcal{M}_{\text{cf}})$. Let $\{X_i\}_{i \in I}$ be any family of objects in $\mathcal{Q} \cap \mathcal{R}$. Assuming that C is compact in \mathcal{M} , we know that the canonical homomorphism

$$(\#10) \quad \sigma : \bigoplus_{i \in I} \text{Hom}_{\mathcal{M}}(C, X_i) \longrightarrow \text{Hom}_{\mathcal{M}}(C, \bigoplus_{i \in I} X_i)$$

is bijective. By definition, σ is given by $(\alpha_i)_{i \in I} \mapsto \sum_{i \in I} \iota_i \alpha_i$ where $\iota_j : X_j \rightarrow \bigoplus_{i \in I} X_i$ is the canonical injection. Recall that in any model category, the class \mathcal{Q} of cofibrant objects is closed under coproducts while the class \mathcal{R} of fibrant objects is closed under products. We have assumed that \mathcal{R} is closed under coproducts in \mathcal{M} , and hence so is the intersection $\mathcal{Q} \cap \mathcal{R}$. The class $\mathcal{Q} \cap \mathcal{W} \cap \mathcal{R}$ is also closed under coproducts in \mathcal{M} . Indeed, \mathcal{R} is closed under coproducts by assumption, and $\mathcal{Q} \cap \mathcal{W}$ is closed under coproducts as it is equal to ${}^\perp \mathcal{R}$, see 2.2. This implies that the coproduct of the family $\{X_i\}_{i \in I}$ computed in $\text{Stab}(\mathcal{M}_{\text{cf}})$ is just $\bigoplus_{i \in I} X_i$, i.e. the coproduct in \mathcal{M} , and the canonical injections are $[\iota_j] : X_j \rightarrow \bigoplus_{i \in I} X_i$, where $[\cdot] : \mathcal{Q} \cap \mathcal{R} \rightarrow \text{Stab}(\mathcal{M}_{\text{cf}})$ denotes the canonical functor. Consequently, for the category $\mathcal{A} = \text{Stab}(\mathcal{M}_{\text{cf}})$, the canonical homomorphism from Definition 4.2 takes the form

$$\begin{aligned} \tau : \bigoplus_{i \in I} \underline{\text{Hom}}_{\mathcal{M}}(C, X_i) &\longrightarrow \underline{\text{Hom}}_{\mathcal{M}}(C, \bigoplus_{i \in I} X_i) && \text{given by} \\ ([\alpha_i])_{i \in I} &\longmapsto \sum_{i \in I} [\iota_i][\alpha_i] = [\sum_{i \in I} \iota_i \alpha_i]. \end{aligned}$$

Here we have used the same notation as in the proof of Proposition 4.1, see (#6). We want to show that τ is an isomorphism. In view of the definitions of τ and

the hom-sets $\underline{\text{Hom}}_{\mathcal{M}}$, this amounts to showing that the isomorphism σ from (#10) restricts to an isomorphism

$$\sigma: \bigoplus_{i \in I} \mathcal{I}(C, X_i) \longrightarrow \mathcal{I}(C, \bigoplus_{i \in I} X_i).$$

This is straightforward to prove and left to the reader. □

Theorem 4.5. *Let \mathcal{M} be a cocomplete abelian category equipped with a hereditary abelian model structure induced by the Hovey triple $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$. Assume that:*

- (1) *The model structure is projective, i.e. one has $\mathcal{R} = \mathcal{M}$.*
- (2) *The cotorsion pair $(\mathcal{Q}, \mathcal{W})$ is generated by a set of compact objects in \mathcal{M} , i.e. there exists a subset $\mathbb{S} \subseteq \mathcal{M}^c$ with $\mathbb{S}^\perp = \mathcal{W}$.*

In this case, the homotopy category of \mathcal{M} is a compactly generated triangulated category; in fact, \mathbb{S} is a set of compact objects in $\text{Ho}(\mathcal{M})$ that generates $\text{Ho}(\mathcal{M})$. In particular,

$$\text{Ho}(\mathcal{M}) = \text{Loc}(\mathbb{S}) \quad \text{and} \quad \text{Ho}(\mathcal{M})^c = \text{Thick}(\mathbb{S}).$$

Proof. First observe that \mathbb{S} is contained in \mathcal{Q} since $\mathbb{S}^\perp = \mathcal{W}$ implies that $\mathbb{S} \subseteq {}^\perp(\mathbb{S}^\perp) = {}^\perp\mathcal{W} = \mathcal{Q}$. Evidently, \mathbb{S} is also contained in $\mathcal{R} = \mathcal{M}$. We have assumed that $\mathbb{S} \subseteq \mathcal{M}^c$, so Lemma 4.4 implies that \mathbb{S} is a set of compact objects in $\text{Ho}(\mathcal{M})$. To see that \mathbb{S} generates $\text{Ho}(\mathcal{M})$, in the sense of Definition 4.3, note that for every $S \in \mathbb{S}$ and $Y \in \mathcal{M}$ one has:

$$\text{Hom}_{\text{Ho}(\mathcal{M})}(S, Y) \cong \text{Hom}_{\text{Ho}(\mathcal{M})}(S, \Sigma(\Sigma^{-1}Y)) \cong \text{Ext}_{\mathcal{M}}^1(S, \mathcal{Q}(\Sigma^{-1}Y)).$$

Indeed, the first isomorphism is trivial and the second follows from Proposition 4.1, where it has been used that every object in \mathcal{M} is a fibrant replacement of itself (since $\mathcal{R} = \mathcal{M}$) and S is a cofibrant replacement of itself (since $\mathbb{S} \subseteq \mathcal{Q}$). Thus, if $\text{Hom}_{\text{Ho}(\mathcal{M})}(S, Y) = 0$ holds for every $S \in \mathbb{S}$, then $\mathcal{Q}(\Sigma^{-1}Y)$ belongs to $\mathbb{S}^\perp = \mathcal{W}$, and hence one has $\mathcal{Q}(\Sigma^{-1}Y) = 0$ in $\text{Ho}(\mathcal{M})$. But in $\text{Ho}(\mathcal{M})$ there is an isomorphism $\mathcal{Q}(\Sigma^{-1}Y) \cong \Sigma^{-1}Y$, and consequently $\Sigma^{-1}Y = 0$ in $\text{Ho}(\mathcal{M})$, which implies that $Y = 0$. This proves that \mathbb{S} generates $\text{Ho}(\mathcal{M})$.

The last assertion is now an immediate consequence of Neeman [29, Thm. 2.1]. □

To parse the next result, recall that the definition of the (co)homology functors from 2.7 only requires the Strong Retraction Property (condition (4) in 2.6). Recall also the functors C_q and K_q from 2.8.

Proposition 4.6. *Assume that Q is Hom-finite, locally bounded and satisfies the Strong Retraction Property, i.e. conditions (1), (2), and (4) in 2.6 hold, and that \mathbb{k} is noetherian. For every $q \in Q$ and $i \geq 0$ the (co)homology functors*

$$\mathbb{H}_{[q]}^i, \mathbb{H}_i^{[q]}: {}_Q A \text{ Mod} \longrightarrow A \text{ Mod}$$

preserve products and filtered colimits. In particular, the left adjoint functor C_q preserves products and right adjoint functor K_q preserves filtered colimits.

Proof. Under the given assumptions [18, Lem. 5.6] shows that the Grothendieck categories ${}_Q \text{Mod}$ and Mod_Q are locally noetherian; in fact, $\{Q(p, -)\}_{p \in Q}$ and $\{Q(-, p)\}_{p \in Q}$ are generating families of noetherian projective objects in these categories. Thus, the noetherian objects $S\langle q \rangle \in {}_Q \text{Mod}$ and $S\{q\} \in \text{Mod}_Q$ from 2.7

have projective resolutions,

$$P_\bullet = \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \text{ in } {}_Q\text{Mod}$$

and $P'_\bullet = \cdots \longrightarrow P'_1 \longrightarrow P'_0 \longrightarrow 0 \text{ in } \text{Mod}_Q,$

where each P_i and each P'_i is a finite direct sum of representable objects, that is, objects of the form $Q(p, -)$ and $Q(-, p)$, respectively. By the Yoneda isomorphisms [26, eq. (3.10)] the functors $\text{Hom}_Q(Q(p, -), ?)$ and $Q(-, p) \otimes_Q ?$ are both naturally isomorphic to the evaluation functor $E_p(?): {}_{Q,A}\text{Mod} \rightarrow {}_A\text{Mod}$ from 2.5, which evidently preserves all limits and colimits. It follows that for every $i \geq 0$ the functors $\text{Hom}_Q(P_i, ?)$ and $P'_i \otimes_Q ?$ preserve all limits and colimits and hence so does the functors $\text{Hom}_Q(P_\bullet, ?)$ and $P'_\bullet \otimes_Q ?$, which goes from ${}_{Q,A}\text{Mod}$ to the category $\text{Ch } A$ of chain complexes of left A -modules. Since homology $H_*: \text{Ch } A \rightarrow {}_A\text{Mod}$ preserves products and filtered colimits, it follows that the functors

$$\mathbb{H}_{[q]}^i = \text{Ext}_Q^i(S\langle q \rangle, ?) = H_{-i} \text{Hom}_Q(P_\bullet, ?)$$

and $\mathbb{H}_i^{[q]} = \text{Tor}_i^Q(S\{q\}, ?) = H_i(P'_\bullet \otimes_Q ?)$

preserve products and filtered colimits too. The last assertion in the proposition now follows immediately as one has $C_q = \mathbb{H}_0^{[q]}$ and $K_q = \mathbb{H}_{[q]}^0$ by definition. \square

Remark 4.7. Let \mathcal{M} be a Grothendieck category. Following Gillespie [13, Dfn. 3.1], an object $F \in \mathcal{M}$ is said to be of *type* FP_∞ if the functors $\text{Ext}_{\mathcal{M}}^i(F, -)$ preserve filtered colimits for all $i \geq 0$. Evidently, every such object is also compact in the sense of Definition 4.2.

As mentioned in the proof above, the category ${}_Q\text{Mod}$ is locally noetherian and the object $S\langle q \rangle$ is noetherian (under the assumptions in Proposition 4.6). Thus [13, Thm. 3.17] shows that $S\langle q \rangle$ is of type FP_∞ , and hence $\mathbb{H}_{[q]}^i = \text{Ext}_Q^i(S\langle q \rangle, ?)$ preserves filtered colimits for every $i \geq 0$. So part of the statement in Proposition 4.6 follows directly from [13].

Corollary 4.8. *Adopt the assumptions from Proposition 4.6. If $M \in {}_A\text{Mod}$ is finitely generated, then $S_q(M)$ is a compact object in ${}_{Q,A}\text{Mod}$ for every $q \in Q$.*

Proof. Fix $q \in Q$. By 2.8 there is a natural isomorphism,

$$(\#11) \quad \text{Hom}_{Q,A}(S_q(M), -) \cong \text{Hom}_A(M, K_q(-)).$$

The functor K_q preserves coproducts by Proposition 4.6, and $\text{Hom}_A(M, -)$ preserves coproducts as M is finitely generated. Thus, the functor in (#11) preserves coproducts. \square

We are now in a position to prove Theorem D from Section 1.

Proof of Theorem D. We will apply Theorem 4.5 to the abelian category $\mathcal{M} = {}_{Q,A}\text{Mod}$ equipped with the projective model structure corresponding to the hereditary Hovey triple

$$(Q, \mathcal{W}, \mathcal{R}) = ({}^\perp \mathcal{E}, \mathcal{E}, \mathcal{M}),$$

see 2.10. It remains to verify condition (2) in Theorem 4.5. We have $\mathbb{S} \subseteq \mathcal{M}^c$ by Corollary 4.8. Furthermore, for every $X \in \mathcal{M} = {}_{Q,A}\text{Mod}$ and $q \in Q$ there are

isomorphisms:

$$\begin{aligned} \text{Ext}_{Q,A}^1(S_q(A), X) &\cong \text{Ext}_{Q,A}^1(S\langle q \rangle \otimes_{\mathbb{k}} A, X) \\ &\cong \text{Ext}_Q^1(S\langle q \rangle, X^{\natural}) \\ &\cong \mathbb{H}_{[q]}^1(X^{\natural}) \\ &\cong \mathbb{H}_{[q]}^1(X)^{\natural}. \end{aligned}$$

Indeed, the 1st isomorphism holds by definition, see 2.8. By [18, Lem. 7.10], the stalk functor $S\langle q \rangle: Q \rightarrow {}_{\mathbb{k}}\text{Mod}$ satisfies $S\langle q \rangle(p) \in {}_{\mathbb{k}}\text{Prj} \subseteq {}_{\mathbb{k}}\text{GPrj}$ for every $p \in Q$. Hence $S\langle q \rangle$ is a Gorenstein projective object in ${}_Q\text{Mod}$ by [18, Thm. 2.7]; in symbols: $S\langle q \rangle \in {}_Q\text{GPrj}$. Thus, the 2nd isomorphism above follows from [18, Lem. 4.3(a)]. The 3rd and 4th isomorphisms hold by [18, Dfn. 7.11 and Rmk. 7.12]. From these isomorphisms the first equivalence below follows; the second equivalence holds by [18, Thm. 7.1]:

$$\begin{aligned} \text{Ext}_{Q,A}^1(S_q(A), X) = 0 \text{ for each } q \in Q &\iff \mathbb{H}_{[q]}^1(X) = 0 \text{ for each } q \in Q \\ &\iff X \in \mathcal{E}. \end{aligned}$$

Consequently, one has $\mathbb{S}^{\perp} = \mathcal{E} = \mathcal{W}$, as desired. □

5. PERFECT OBJECTS IN $\mathcal{D}_Q(A)$

In this section, we introduce perfect objects and compare them to compact objects.

Definition 5.1. The *support* of an object X in ${}_Q, A\text{Mod}$ is defined as

$$\text{supp } X = \{q \in Q \mid X(q) \neq 0\};$$

it is a subset of $\text{Ob}(Q)$. We say that X has *finite support* if $\text{supp } X$ is a finite set.

The next lemma follows immediately from the definition.

Lemma 5.2. *For every short exact sequence $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ in ${}_Q, A\text{Mod}$ one has*

$$\text{supp } X = \text{supp } X' \cup \text{supp } X''.$$

Thus, the class of objects in ${}_Q, A\text{Mod}$ with finite support is a Serre subcategory. □

Definition 5.3. We introduce the following notions.

- An object K in ${}_Q, A\text{Mod}$ is called *strictly perfect* if it has finite support and $K(q)$ is a finitely generated projective left A -module for every $q \in Q$. We set

$${}_Q, A\mathcal{K} = \{K \in {}_Q, A\text{Mod} \mid K \text{ is strictly perfect}\}.$$

- Assume Setup 2.9. An object X in $\mathcal{D}_Q(A)$ is called *perfect* if it has a strictly perfect cofibrant replacement in the projective model structure on ${}_Q, A\text{Mod}$; that is:

$$X \text{ is perfect} \iff \begin{cases} \text{There is an isomorphism } X \cong K \text{ in } \mathcal{D}_Q(A) \\ \text{for some object } K \in {}^{\perp}\mathcal{E} \cap {}_Q, A\mathcal{K}. \end{cases}$$

We set

$$\mathcal{D}_Q^{\text{perf}}(A) = \{X \in \mathcal{D}_Q(A) \mid X \text{ is perfect}\}.$$

Example 5.4. As mentioned in Section 1: In the special case of chain complexes, strictly perfect objects are bounded complexes of finitely generated modules.

Beware that a strictly perfect object need not be perfect; see Theorem A.3 for a counterexample. On the positive side, one does have Theorem B (part (c)) from Section 1, which will be proved later in this section.

The next result follows immediately from Lemma 5.2 and the definition, 5.3, of strictly perfect objects.

Lemma 5.5. *Let $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ be an exact sequence in ${}_{Q,A}\text{Mod}$ with $X'' \in {}_{Q,A}\mathcal{K}$. Then one has $X' \in {}_{Q,A}\mathcal{K}$ if and only if $X \in {}_{Q,A}\mathcal{K}$. \square*

Construction 5.6. Consider the adjoint triple from 2.5. We introduce the notation:

- ε_q for the counit of the adjunction (F_q, E_q) .
- η_q for the unit of the adjunction (E_q, G_q) .

Consider the endofunctors on the category ${}_{Q,A}\text{Mod}$ given by

$$\mathbb{F} = \bigoplus_{q \in Q} F_q E_q \quad \text{and} \quad \mathbb{G} = \prod_{q \in Q} G_q E_q.$$

By the universal properties of coproducts and products, there exist unique morphisms ε and η in ${}_{Q,A}\text{Mod}$ that make the diagrams

$$\begin{array}{ccc}
 \begin{array}{c} F_p E_p \\ \downarrow \iota_p \\ \mathbb{F} = \bigoplus_{q \in Q} F_q E_q \end{array} & \begin{array}{c} \searrow \varepsilon_p \\ \\ \dots \varepsilon \end{array} & \begin{array}{c} \\ \\ \text{Id} \end{array} \\
 \text{(#12)} & \text{and} & \\
 \begin{array}{c} \\ \\ \text{Id} \end{array} & \begin{array}{c} \nearrow \eta_p \\ \\ \dots \eta \end{array} & \begin{array}{c} G_p E_p \\ \downarrow \pi_p \\ \mathbb{G} = \prod_{q \in Q} G_q E_q \end{array}
 \end{array}$$

commutative for every $p \in Q$; here ι_p is the injection, π_p is the projection, and Id denotes the identity functor on ${}_{Q,A}\text{Mod}$. We set

$$\mathbb{K} = \text{Ker } \varepsilon \quad \text{and} \quad \mathbb{C} = \text{Cok } \eta,$$

which are also endofunctors on ${}_{Q,A}\text{Mod}$.

Proposition 5.7. *With the notation from Construction 5.6 one has short exact sequences of endofunctors on ${}_{Q,A}\text{Mod}$,*

$$\mathfrak{F} = 0 \longrightarrow \mathbb{K} \longrightarrow \mathbb{F} \xrightarrow{\varepsilon} \text{Id} \longrightarrow 0 \quad \text{and} \quad \mathfrak{G} = 0 \longrightarrow \text{Id} \xrightarrow{\eta} \mathbb{G} \longrightarrow \mathbb{C} \longrightarrow 0.$$

For every object X in ${}_{Q,A}\text{Mod}$, the short exact sequences $\mathfrak{F}(X)$ and $\mathfrak{G}(X)$ in ${}_{Q,A}\text{Mod}$ are objectwise split (see Definition 3.8).

Proof. We only prove the assertions about the sequence \mathfrak{F} as the sequence \mathfrak{G} can be handled similarly. For every X in ${}_{Q,A}\text{Mod}$, Construction 5.6 yields an exact sequence

$$0 \longrightarrow \mathbb{K}(X) \longrightarrow \mathbb{F}(X) \xrightarrow{\varepsilon^X} X.$$

Here we have used the notation ε^X instead of $\varepsilon(X)$. We must prove that ε^X is epic, i.e. the morphism $\varepsilon^X(p)$ in ${}_A\text{Mod}$ is epic for every $p \in Q$. We will even show that $\varepsilon^X(p)$ is split epic, which also takes care of the last assertion in the lemma. By evaluating the leftmost diagram in (#12) at X and applying the functor E_p to the resulting diagram, one gets

$$\text{(#13)} \quad E_p(\varepsilon^X) \circ E_p(\iota_p^X) = E_p(\varepsilon_p^X).$$

Let ζ_p be the unit of the adjunction (F_p, E_p) . By the triangle (or the zigzag) identities for adjoint functors, see [28, Chap. IV.1, Thm. 1(ii)], the composite

$$E_p \xrightarrow{\zeta_p E_p} E_p F_p E_p \xrightarrow{E_p \varepsilon_p} E_p$$

is the identity transformation on E_p , so the morphism $\zeta_p^{E_p(X)}$ is a right inverse of $E_p(\varepsilon_p^X)$. Combined with identity (#13), it follows that the composite $E_p(\iota_p^X) \circ \zeta_p^{E_p(X)}$ is a right inverse of the morphism $E_p(\varepsilon^X) = \varepsilon^X(p)$, as desired. \square

Lemma 5.8. *Assume that Q is Hom-finite and locally bounded, i.e. Q satisfies conditions (1) and (2) in 2.6. For every object $X \in {}_Q, A\text{Mod}$ the following assertions hold.*

- (a) *If X has finite support, then $\mathbb{F}(X)$, $\mathbb{G}(X)$, $\mathbb{K}(X)$, and $\mathbb{C}(X)$ have finite support.*
- (b) *If X belongs to ${}_Q, A\mathcal{K}$, then $\mathbb{F}(X)$, $\mathbb{G}(X)$, $\mathbb{K}(X)$, and $\mathbb{C}(X)$ belong to ${}_Q, A\mathcal{K}$.*
- (c) *If X belongs to ${}_Q, A\mathcal{K}$, then $\mathbb{F}(X)$ is a finitely presented projective object in ${}_Q, A\text{Mod}$.*

Proof. Recall from Proposition 5.7 that there are short exact sequences,

$$\begin{aligned} \mathfrak{F}(X) = 0 &\longrightarrow \mathbb{K}(X) \longrightarrow \mathbb{F}(X) \longrightarrow X \longrightarrow 0 \\ \text{and } \mathfrak{G}(X) = 0 &\longrightarrow X \longrightarrow \mathbb{G}(X) \longrightarrow \mathbb{C}(X) \longrightarrow 0. \end{aligned}$$

(a): Assume that X has finite support. By the short exact sequences above and by Lemma 5.2 it suffices to argue that the objects $\mathbb{F}(X)$ and $\mathbb{G}(X)$ have finite support. By Construction 5.6 and by the definition of support, see 5.1, there are for each $p \in Q$ equalities:

$$\begin{aligned} \text{(#14)} \quad \mathbb{F}(X)(p) &= \bigoplus_{q \in Q} F_q(X(q))(p) = \bigoplus_{q \in \text{supp } X} F_q(X(q))(p), \\ \mathbb{G}(X)(p) &= \prod_{q \in Q} G_q(X(q))(p) = \prod_{q \in \text{supp } X} G_q(X(q))(p). \end{aligned}$$

Now fix $q \in \text{supp } X$ and recall from 2.5 that one has

$$\begin{aligned} \text{(#15)} \quad F_q(X(q))(p) &= Q(q, p) \otimes_{\mathbb{k}} X(q), \\ G_q(X(q))(p) &= \text{Hom}_{\mathbb{k}}(Q(p, q), X(q)). \end{aligned}$$

We see that if $F_q(X(q))(p) \neq 0$ then p must be in the set $N_+(q)$, and if $G_q(X(q))(p) \neq 0$ then p must be in the set $N_-(q)$; the sets $N_{\pm}(q)$ are defined in condition (2) in 2.6. Hence there are inclusions:

$$\text{supp } \mathbb{F}(X) \subseteq \bigcup_{q \in \text{supp } X} N_+(q) \quad \text{and} \quad \text{supp } \mathbb{G}(X) \subseteq \bigcup_{q \in \text{supp } X} N_-(q).$$

By assumption, the set $\text{supp } X$ is finite, and so are the sets $N_{\pm}(q)$ for each q as Q is locally bounded. Thus, the inclusions above show that $\mathbb{F}(X)$ and $\mathbb{G}(X)$ have finite support.

(b): Assume that X belongs to ${}_Q, A\mathcal{K}$. From part (a) we know that $\mathbb{F}(X)$, $\mathbb{G}(X)$, $\mathbb{K}(X)$, and $\mathbb{C}(X)$ have finite support, so it remains to show that for each $p \in Q$ the left A -modules $\mathbb{F}(X)(p)$, $\mathbb{G}(X)(p)$, $\mathbb{K}(X)(p)$, and $\mathbb{C}(X)(p)$ are finitely generated projective. Proposition 5.7 shows that for each $p \in Q$ the sequences $\mathfrak{F}(X)(p)$ and $\mathfrak{G}(X)(p)$ are split exact in ${}_A\text{Mod}$, so it suffices to see that $\mathbb{F}(X)(p)$ and $\mathbb{G}(X)(p)$ are finitely generated projective. Fix $p \in Q$. As Q is Hom-finite, the \mathbb{k} -modules $Q(q, p)$ and $Q(p, q)$ are finitely generated projective for every $q \in Q$. As $X(q)$ is a finitely generated projective left A -module for every $q \in Q$, it follows from (#15)

that $F_q(X(q))(p)$ and $G_q(X(q))(p)$ are finitely generated projective left A -modules for every $q \in Q$. Since $\text{supp } X$ is a finite set, (#14) now shows that $\mathbb{F}(X)(p)$ and $\mathbb{G}(X)(p)$ are finite direct sums of finitely generated projective left A -modules; hence $\mathbb{F}(X)(p)$ and $\mathbb{G}(X)(p)$ are finitely generated projective left A -modules.

(c): From (#14) we know that $\mathbb{F}(X)$ is the finite direct sum $\bigoplus_{q \in \text{supp } X} F_q(X(q))$; and by [18, Lem. 3.11] each object $F_q(X(q))$ is finitely presented and projective in ${}_{Q,A}\text{Mod}$. \square

Proposition 5.9. *Assume that Q is Hom-finite and locally bounded, i.e. Q satisfies conditions (1) and (2) in 2.6. Every strictly perfect object in ${}_{Q,A}\text{Mod}$ is of type FP_∞ (see Remark 4.7); in particular, there is an inclusion:*

$${}_{Q,A}\mathcal{K} \subseteq ({}_{Q,A}\text{Mod})^c.$$

Proof. Consider the endofunctors \mathbb{F} and \mathbb{K} on ${}_{Q,A}\text{Mod}$ from Construction 5.6. For $i \geq 0$ we denote by \mathbb{K}^i the i -fold composite $\mathbb{K} \circ \dots \circ \mathbb{K}$ with the convention that $\mathbb{K}^0 = \text{Id}$ is the identity functor. Let $X \in {}_{Q,A}\text{Mod}$. Pasting together the short exact sequences,

$$\mathfrak{F}(\mathbb{K}^i(X)) = 0 \longrightarrow \mathbb{K}^{i+1}(X) \longrightarrow \mathbb{F}(\mathbb{K}^i(X)) \longrightarrow \mathbb{K}^i(X) \longrightarrow 0 \quad (i \geq 0)$$

coming from Proposition 5.7, we get a long exact sequence,

$$\text{(#16)} \quad \dots \longrightarrow \mathbb{F}(\mathbb{K}^2(X)) \longrightarrow \mathbb{F}(\mathbb{K}(X)) \longrightarrow \mathbb{F}(X) \longrightarrow X \longrightarrow 0.$$

If $X \in {}_{Q,A}\mathcal{K}$, then successive applications of Lemma 5.8(b) show that $\mathbb{K}^i(X) \in {}_{Q,A}\mathcal{K}$ for every $i \geq 0$, and hence each $\mathbb{F}(\mathbb{K}^i(X))$ is finitely presented and projective by Lemma 5.8(c). Thus (#16) is a projective resolution of X by finitely presented (and projective) objects, and it follows that X is of type FP_∞ . \square

Corollary 5.10. *Assume Setup 2.9. Every perfect object in $\mathcal{D}_Q(A)$ is compact, that is:*

$$\mathcal{D}_Q^{\text{perf}}(A) \subseteq \mathcal{D}_Q(A)^c.$$

Proof. The class of compact objects in $\mathcal{D}_Q(A)$ is closed under isomorphisms, so to prove the inclusion $\mathcal{D}_Q^{\text{perf}}(A) \subseteq \mathcal{D}_Q(A)^c$ it suffices by Definition 5.3 to argue that every object $K \in {}^\perp \mathcal{E} \cap {}_{Q,A}\mathcal{K}$ is compact in $\mathcal{D}_Q(A)$. Recall from 2.10 that $\mathcal{D}_Q(A)$ is the homotopy category of ${}_{Q,A}\text{Mod}$ equipped with the hereditary abelian model structure where $\mathcal{Q} = {}^\perp \mathcal{E}$ is the class of cofibrant objects and every object is fibrant. Thus every object $K \in {}^\perp \mathcal{E} \cap {}_{Q,A}\mathcal{K}$ is both cofibrant and fibrant, and by Proposition 5.9 it is also compact in ${}_{Q,A}\text{Mod}$. Thus, Lemma 4.4 implies that K is also compact in $\text{Ho}({}_{Q,A}\text{Mod}) = \mathcal{D}_Q(A)$. \square

Our next goal is to show that $\mathcal{D}_Q^{\text{perf}}(A)$ is even a triangulated subcategory of $\mathcal{D}_Q(A)^c$.

Proposition 5.11. *Assume Setup 2.9. Consider for $X \in {}_{Q,A}\text{Mod}$ the exact sequences,*

$$\begin{aligned} \mathfrak{F}(X) = 0 &\longrightarrow \mathbb{K}(X) \longrightarrow \mathbb{F}(X) \longrightarrow X \longrightarrow 0 \\ \text{and } \mathfrak{G}(X) = 0 &\longrightarrow X \longrightarrow \mathbb{G}(X) \longrightarrow \mathbb{C}(X) \longrightarrow 0 \end{aligned}$$

in ${}_{Q,A}\text{Mod}$ from Proposition 5.7. The following assertions hold:

- (a) *If $X \in {}^\perp \mathcal{E}$, then one has $\mathbb{F}(X), \mathbb{G}(X) \in {}_{Q,A}\text{Prj}$ and $\mathbb{K}(X), \mathbb{C}(X) \in {}^\perp \mathcal{E}$, and hence:*

- The sequences $\mathfrak{F}(X)$ and $\mathfrak{G}(X)$ are conflations in the exact category ${}^\perp \mathcal{E}$.
 - In the triangulated stable category ${}^\perp \mathcal{E} / {}_{Q,A} \text{Prj}$, whose translation functor we denote by Σ , there are isomorphisms $\Sigma X \cong \mathbb{C}(X)$ and $\Sigma^{-1} X \cong \mathbb{K}(X)$.
- (b) If $X \in \mathcal{E}^\perp$, then one has $\mathbb{F}(X), \mathbb{G}(X) \in {}_{Q,A} \text{Inj}$ and $\mathbb{K}(X), \mathbb{C}(X) \in \mathcal{E}^\perp$, and hence:
- The sequences $\mathfrak{F}(X)$ and $\mathfrak{G}(X)$ are conflations in the exact category \mathcal{E}^\perp .
 - In the triangulated stable category $\mathcal{E}^\perp / {}_{Q,A} \text{Inj}$, whose translation functor we denote by Σ , there are isomorphisms $\Sigma X \cong \mathbb{C}(X)$ and $\Sigma^{-1} X \cong \mathbb{K}(X)$.

Proof. (a): Assume that X belongs to ${}^\perp \mathcal{E}$. For every $q \in Q$ the left A -module $E_q(X) = X(q)$ is projective by Proposition 3.2(a), and hence $F_q E_q(X)$ is a projective object in ${}_{Q,A} \text{Mod}$ by [18, Lem. 3.11]. Since the class ${}_{Q,A} \text{Prj}$ is closed under coproducts, it follows that the object $\mathbb{F}(X) = \bigoplus_{q \in Q} F_q E_q(X)$ belongs to ${}_{Q,A} \text{Prj}$. Lemma 3.4 and Proposition 3.7 now imply that also $\mathbb{G}(X) = \prod_{q \in Q} G_q E_q(X)$ belongs to ${}_{Q,A} \text{Prj}$. In particular, the objects $X, \mathbb{F}(X)$, and $\mathbb{G}(X)$ are in ${}^\perp \mathcal{E}$. As the exact sequences $\mathfrak{F}(X)$ and $\mathfrak{G}(X)$ are objectwise split by Proposition 5.7, it now follows from Theorem 3.9(a,b) that $\mathbb{K}(X)$ and $\mathbb{C}(X)$ are in ${}^\perp \mathcal{E}$.

As the class ${}^\perp \mathcal{E}$ is closed under extensions in the abelian category ${}_{Q,A} \text{Mod}$, it naturally inherits the structure of an exact category where the conflations are short exact sequences $0 \rightarrow Y' \rightarrow Y \rightarrow Y'' \rightarrow 0$ in ${}_{Q,A} \text{Mod}$ with $Y', Y, Y'' \in {}^\perp \mathcal{E}$. This explains the first bullet point. The second bullet point is evident from the general construction/definition of the translation functor in the stable category of a Frobenius category; see Happel [15, Chap. I§2].

(b): Dual to the proof of part (a). □

Theorem 5.12. *Assume Setup 2.9. The subcategory $\mathcal{D}_Q^{\text{perf}}(A)$ of $\mathcal{D}_Q(A)$ is triangulated.*

Proof. Write $\mathcal{T} = {}^\perp \mathcal{E} / {}_{Q,A} \text{Prj}$ for the stable category of the Frobenius category ${}^\perp \mathcal{E}$. Set

$$\mathcal{K}' = {}^\perp \mathcal{E} \cap {}_{Q,A} \mathcal{K}.$$

By [18, Thm. 6.5] there is an equivalence of triangulated categories, $\mathcal{D}_Q(A) \simeq \mathcal{T}$; and via this equivalence, the full subcategory $\mathcal{D}_Q^{\text{perf}}(A) \subseteq \mathcal{D}_Q(A)$ corresponds, in view of Definition 5.3, to the full subcategory $\mathcal{T}^{\text{perf}} \subseteq \mathcal{T}$ defined by

$$\mathcal{T}^{\text{perf}} = \left\{ X \in \mathcal{T} \mid \begin{array}{l} \text{There is an isomorphism } X \cong K \\ \text{in } \mathcal{T} \text{ for some object } K \in \mathcal{K}' \end{array} \right\}.$$

Thus, it suffices to argue that $\mathcal{T}^{\text{perf}}$ is a triangulated subcategory of \mathcal{T} . We will verify the requirements in Neeman [30, Dfn. 1.5.1].

As \mathcal{K}' is clearly an additive subcategory of ${}^\perp \mathcal{E} \subseteq {}_{Q,A} \text{Mod}$, it follows that $\mathcal{T}^{\text{perf}}$ is an additive subcategory of \mathcal{T} . By definition, $\mathcal{T}^{\text{perf}}$ is closed under isomorphisms in \mathcal{T} .

Let Σ denote the translation functor in the triangulated category \mathcal{T} . We must show that one has $\Sigma(\mathcal{T}^{\text{perf}}) = \mathcal{T}^{\text{perf}}$; equivalently, that the following two inclusions hold:

$$\Sigma(\mathcal{T}^{\text{perf}}) \subseteq \mathcal{T}^{\text{perf}} \quad \text{and} \quad \Sigma^{-1}(\mathcal{T}^{\text{perf}}) \subseteq \mathcal{T}^{\text{perf}}.$$

By the definition of $\mathcal{T}^{\text{perf}}$, it suffices to prove the inclusions:

$$\Sigma(\mathcal{K}') \subseteq \mathcal{T}^{\text{perf}} \quad \text{and} \quad \Sigma^{-1}(\mathcal{K}') \subseteq \mathcal{T}^{\text{perf}}.$$

To establish these inclusions, let X be an object in \mathcal{K}' . By part (a) in Proposition 5.11 we have $\Sigma X \cong \mathbb{C}(X)$ and $\Sigma^{-1}X \cong \mathbb{K}(X)$, so it suffices to argue that the objects $\mathbb{C}(X)$ and $\mathbb{K}(X)$ belong to \mathcal{K}' , but this follows from Lemma 5.8(b) and Proposition 5.11(a).

It remains to prove that if $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$ is a distinguished triangle in \mathcal{T} with $X, Y \in \mathcal{T}^{\text{perf}}$, then one has $Z \in \mathcal{T}^{\text{perf}}$. Without loss of generality, we can assume that X and Y belong to \mathcal{K}' . Let $[\alpha]: X \rightarrow Y$ be the morphism from X to Y in the given distinguished triangle; here $\alpha: X \rightarrow Y$ is a morphism in ${}_{Q,A}\text{Mod}$ (between objects $X, Y \in \mathcal{K}' \subseteq {}^\perp \mathcal{E}$) and $[\alpha]$ denote its class (image) in the stable category \mathcal{T} . Consider in the abelian category ${}_{Q,A}\text{Mod}$ the pushout diagram below where the upper exact sequence comes from Proposition 5.7:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X & \longrightarrow & \mathbb{G}(X) & \longrightarrow & \mathbb{C}(X) \longrightarrow 0 \\
 & & \alpha \downarrow & \text{(pushout)} & \downarrow & & \parallel \\
 (\#17) & & 0 & \longrightarrow & Y & \longrightarrow & P \longrightarrow \mathbb{C}(X) \longrightarrow 0.
 \end{array}$$

Recall from Proposition 5.11(a) that $\mathbb{C}(X) \cong \Sigma X$ in \mathcal{T} . The diagram $X \xrightarrow{[\alpha]} Y \rightarrow P \rightarrow \Sigma X$ is a distinguished triangle in \mathcal{T} (called a *standard* triangle), see Happel [15, Chap. I§2, (2.5)], and hence there exists by [30, Rmk. 1.1.21] an isomorphism $Z \cong P$ in \mathcal{T} . Thus, to show $Z \in \mathcal{T}^{\text{perf}}$ it suffices to argue that $P \in \mathcal{K}'$. As $X \in \mathcal{K}'$ we have, as above, $\mathbb{C}(X) \in \mathcal{K}'$. Since also $Y \in \mathcal{K}'$ we can now apply Lemma 5.5 to the bottom exact sequence in (#17) to conclude that $P \in \mathcal{K}'$, as desired. \square

We are now ready to show Theorem C from Section 1.

Proof of Theorem C. Theorem 5.12 shows that $\mathcal{D}_Q^{\text{perf}}(A)$ is a triangulated subcategory of $\mathcal{D}_Q(A)$. And it is a general fact that the subcategory $\mathcal{D}_Q(A)^c$ of compact objects is triangulated and thick, see [3, Property 6.3] or more generally [30, Lem. 4.2.4, Rmk. 4.2.6, and Dfn. 4.2.7]. Corollary 5.10 shows that there is always an inclusion $\mathcal{D}_Q^{\text{perf}}(A) \subseteq \mathcal{D}_Q(A)^c$.

For each $q \in Q$ the object $S_q(A)$ is evidently strictly perfect, see Definition 5.3, and it also belongs to ${}^\perp \mathcal{E}$ by Proposition 3.2(a). Thus, $S_q(A)$ belongs to $\mathcal{D}_Q^{\text{perf}}(A)$, again by Definition 5.3. By Theorem D in Section 1 (proved in Section 4) we have

$$\mathcal{D}_Q(A)^c = \text{Thick}(\{S_q(A) \mid q \in Q\}).$$

It follows that if the subcategory $\mathcal{D}_Q^{\text{perf}}(A)$ is thick, then one has $\mathcal{D}_Q(A)^c \subseteq \mathcal{D}_Q^{\text{perf}}(A)$, and hence equality holds, i.e. $\mathcal{D}_Q(A)^c = \mathcal{D}_Q^{\text{perf}}(A)$. Conversely, if this equality holds, then the subcategory $\mathcal{D}_Q^{\text{perf}}(A)$ must be thick, as this is true for $\mathcal{D}_Q(A)^c$. \square

The Strong Retraction Property allows one to define a notion of cycles.

Definition 5.13. Assume that Q satisfies the Strong Retraction Property, i.e. condition (4) in 2.6 holds. A *cycle* in Q (more precisely, a *τ -cycle*, where τ is the pseudo-radical of Q) is a sequence of $n \geq 1$ morphisms in Q ,

$$q_1 \xrightarrow{g_1} q_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} q_n \xrightarrow{g_n} q_{n+1},$$

where $q_1 = q_{n+1}$ and $0 \neq g_i \in \mathfrak{r}$ for every $i = 1, \dots, n$. The number n is called the *length* of the cycle. A cycle of length $n = 1$ is called a *loop*.

Recall from 2.8 the functor K_q . Let $X \in {}_Q, A \text{Mod}$. As shown in [18, Prop. 7.18] the left A -module $K_q(X)$ is isomorphic to $\bigcap_{g \in \mathfrak{r}(q,*)} \text{Ker } X(g)$ where the intersection is taken over all morphisms $g \in \mathfrak{r}$ with domain q . In particular, $K_q(X)$ can be seen as a submodule of $X(q)$.

Lemma 5.14. *Assume that Q satisfies the Strong Retraction Property, i.e. condition (4) in 2.6 holds. Assume furthermore that Q has no cycles. If an object $X \neq 0$ in ${}_Q, A \text{Mod}$ has finite support, then there exists $q \in \text{supp } X$ with $K_q(X) = X(q)$.*

Proof. Suppose (for contradiction) that $K_q(X) \neq X(q)$ holds for every $q \in \text{supp } X$. Since X is non-zero, it has non-empty support, so we can choose an element $q_1 \in \text{supp } X$. By assumption we have $K_{q_1}(X) \neq X(q_1)$; this means that there exists some $g_1: q_1 \rightarrow q_2$ in \mathfrak{r} with $X(g_1) \neq 0$; in particular, $g_1 \neq 0$ and $q_2 \in \text{supp } X$. Note that $q_1 \neq q_2$ as Q has no cycles. By assumption we have $K_{q_2}(X) \neq X(q_2)$, so as before there exists some $g_2: q_2 \rightarrow q_3$ in \mathfrak{r} with $X(g_2) \neq 0$; in particular, $g_2 \neq 0$ and $q_3 \in \text{supp } X$. Note that $q_3 \neq q_1$ and $q_3 \neq q_2$ as Q has no cycles. Continuing in this manner, we construct an infinite sequence,

$$q_1 \xrightarrow{g_1} q_2 \xrightarrow{g_2} q_3 \xrightarrow{g_3} \dots,$$

of morphisms in \mathfrak{r} where each $X(g_i)$ is non-zero and q_1, q_2, q_3, \dots are different objects in $\text{supp } X$. However, this contradicts the assumption that X has finite support. \square

Remark 5.15. A similar argument shows that if an object $X \neq 0$ in ${}_Q, A \text{Mod}$ has finite support, then there also exists some $p \in \text{supp } X$ with $C_p(X) = X(p)$.

We now prove Theorem B from Section 1.

Proof of Theorem B. (b): If A has finite left global dimension, the inclusion ${}_Q, A \mathcal{K} \subseteq {}^\perp \mathcal{E}$ follows from Definition 5.3 and Theorem E from Section 1 (proved in Section 3).

Next consider the case where Q has no cycles (and A is arbitrary). Let $X \in {}_Q, A \mathcal{K}$; we must show that X belongs to ${}^\perp \mathcal{E}$. We argue by induction on the cardinality $n = |\text{supp } X|$ of the support of X . If $n = 0$, then $\text{supp } X = \emptyset$ and hence $X = 0 \in {}^\perp \mathcal{E}$. Now assume that $n > 0$ and that every $Y \in {}_Q, A \mathcal{K}$ with $|\text{supp } Y| < n$ belongs to ${}^\perp \mathcal{E}$. By Lemma 5.14 there exists $q \in \text{supp } X$ with $K_q(X) = X(q)$. Let ε_q be the counit of the adjunction (S_q, K_q) from 2.8 and note that the morphism

$$S_q(X(q)) = S_q K_q(X) \xrightarrow{\varepsilon_q^X} X$$

is monic as $\varepsilon_q^X(q): X(q) \rightarrow X(q)$ is the identity map and $\varepsilon_q^X(p): 0 \rightarrow X(p)$ is the zero map for $p \neq q$. Thus there is a short exact sequence in ${}_Q, A \text{Mod}$,

$$0 \longrightarrow S_q(X(q)) \longrightarrow X \longrightarrow Y \longrightarrow 0,$$

where $Y(q) = 0$ and $Y(p) = X(p)$ for $p \neq q$. It follows that $Y \in {}_Q, A \mathcal{K}$ with $|\text{supp } Y| = n - 1$, and hence Y belongs to ${}^\perp \mathcal{E}$ by the induction hypothesis. As the left A -module $X(q)$ is (finitely generated and) projective, one has $S_q(X(q)) \in {}^\perp \mathcal{E}$ by Proposition 3.2(a). As the class ${}^\perp \mathcal{E}$ is closed under extensions, see Theorem 3.9(a), we conclude that X is in ${}^\perp \mathcal{E}$.

(c): Having established the inclusion ${}_{Q,A}\mathcal{K} \subseteq {}^\perp \mathcal{E}$, the assertion follows directly from the definition, 5.3, of perfect objects.

(a): By part (c), every strictly perfect object is perfect in $\mathcal{D}_Q(A)$ and every such object is compact in $\mathcal{D}_Q(A)$ by Corollary 5.10. \square

We end this paper with an appendix that contains a proof of Theorem A from Section 1. In fact, we prove the more general Theorem A.3, which shows that the conclusions in parts (a)–(c) of Theorem B (from Section 1) may fail without further assumptions on the category Q or the ring A (like conditions (1) or (2) of that theorem).

APPENDIX A. AN EXAMPLE OF A STRICTLY PERFECT NON-COMPACT OBJECT

Let \mathbb{k} be any commutative, noetherian, and hereditary ring (e.g. \mathbb{k} is a field or $\mathbb{k} = \mathbb{Z}$) as in Setup 2.9, and consider the commutative noetherian \mathbb{k} -algebra

$$A = \mathbb{k}[X, Y]/(X^2, XY).$$

We write x and y for the images of X and Y in the quotient ring A . Note that in A one has $x^2 = xy = 0$; moreover, A is a free \mathbb{k} -module with basis $\{1, x, y, y^2, y^3, \dots\}$.

We consider the Jordan quiver; it has a single vertex $(*)$ and a single loop (ε) :

$$\Gamma : \quad * \begin{array}{c} \curvearrowright \\ \varepsilon \end{array}$$

equipped with the relation $\varepsilon^2 = 0$. Let Q be the path category of Γ over \mathbb{k} modulo the ideal generated by ε^2 ; in symbols: $Q = \mathbb{k}\Gamma/(\varepsilon^2)$. Clearly there is an equivalence of categories,

$${}_{Q,A}\text{Mod} \simeq \text{Diff}(A),$$

where $\text{Diff}(A)$ denotes the category of differential A -modules. An object in $\text{Diff}(A)$ is a pair (X, d) where X is an A -module and d is an endomorphism of X with $d^2 = 0$.

Remark A.1. As the path algebra of Γ over \mathbb{k} modulo the ideal generated by ε^2 is the ring $\mathbb{k}[\varepsilon]/(\varepsilon^2)$ of dual numbers over \mathbb{k} , the category ${}_{Q,A}\text{Mod}$ is also equivalent to the category of modules over $A[\varepsilon]/(\varepsilon^2)$. However, we will work with the category $\text{Diff}(A)$ instead.

A.2. *It is easily seen that Q satisfies all the assumptions in 2.6 and the ideal $\mathfrak{r} = (\varepsilon)$ generated by ε is a pseudo-radical in Q with $\mathfrak{r}^2 = 0$. Hence we are in the setting of Setup 2.9 so the results and definitions from 2.10 apply. The category Q has only one object, which we denote by “ $*$ ”. In this case the functors F_* and S_* from 2.5 and 2.8 are given by*

$$F_*(M) = (M^2, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}) \quad \text{and} \quad S_*(M) = (M, 0)$$

for $M \in {}_A\text{Mod}$; elements in M^2 are viewed as column vectors. The stalk differential module $S(*) = S_*(\mathbb{k}) = (\mathbb{k}, 0)$ in $\text{Diff}(\mathbb{k})$, see 2.7, has the following projective resolution:

$$\dots \xrightarrow{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} F_*(\mathbb{k}) \xrightarrow{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} F_*(\mathbb{k}) \xrightarrow{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} F_*(\mathbb{k}) \xrightarrow{[1 \ 0]} S(*) \longrightarrow 0 .$$

Let (X, d) be in $\text{Diff}(A)$. Applying the functor $\text{Hom}_{\text{Diff}(\mathbb{k})}(-, (X, d))$ to the non-augmented version of the projective resolution above, and using the natural isomorphism

$$\text{Hom}_{\text{Diff}(\mathbb{k})}(F_*(\mathbb{k}), (X, d)) \cong \text{Hom}_{\mathbb{k}}(\mathbb{k}, X) \cong X$$

from 2.5, one gets the complex $0 \rightarrow X \xrightarrow{d} X \xrightarrow{d} X \xrightarrow{d} \dots$. It follows that the cohomology $\mathbb{H}_{[*]}^i(X, d)$ for $i > 0$, see 2.7, is the ordinary (co)homology $\mathbb{H}(X, d) = \text{Ker } d / \text{Im } d$ for differential modules. The same is true for homology, i.e. $\mathbb{H}_i^{[*]} = \mathbb{H}$ for every $i > 0$. Thus:

- A differential A -module (X, d) is exact in the sense of [18, Dfn. 4.1] (see also 2.10) if and only if $\mathbb{H}(X, d) = 0$, that is, $\text{Ker } d = \text{Im } d$. This follows from [18, Thm. 7.1].
- A morphism φ of differential A -modules is a weak equivalence in the projective (or injective) model structure on $\text{Diff}(A)$, see [18, Thm. 6.1 and Prop. 6.3] (and 2.10) if and only if $\mathbb{H}(\varphi)$ is an isomorphism. This follows from [18, Thm. 7.2].

As $x^2 = 0$ holds in A the pair (A, x) is a differential A -module. The goal of this appendix is to prove the next result; it contains Theorem A from Section 1 and shows that the conclusions in parts (a)–(c) of Theorem B (from Section 1) may fail without further assumptions on Q or A . Note that the category Q under investigation in this appendix has a loop, and the ring A has infinite global dimension, so neither assumption (1) nor (2) in Theorem B is satisfied in this case.

Theorem A.3. *Let the rings \mathbb{k} and A be as above. The following assertions hold.*

- (A, x) is a strictly perfect object in $\text{Diff}(A)$.
- (A, x) does not belong to ${}^\perp \mathcal{E}$.
- (A, x) is neither perfect nor compact in $\text{Ho}(\text{Diff}(A))$.

As stated in part (b) of the theorem above, (A, x) does not belong to ${}^\perp \mathcal{E}$; that is, (A, x) is not a cofibrant object in the projective model structure on $\text{Diff}(A)$. In Proposition A.13 we give a cofibrant replacement of (A, x) ; to construct it we need Zeckendorf expansions, which we now explain.

A.4. *Consider the Fibonacci numbers $F_{-2}, F_{-1}, F_0, F_1, F_2, F_3, \dots$ defined by*

$$F_{-2} = 0, \quad F_{-1} = 1 \quad \text{and} \quad F_i = F_{i-1} + F_{i-2} \quad \text{for } i \geq 0;$$

that is $F_0 = 1, F_1 = 2, F_2 = 3, F_3 = 5, F_4 = 8, F_5 = 13, F_6 = 21, F_7 = 34$ etc.

Zeckendorf's theorem [35] (see also Dekking [5]) asserts that for every $n \in \mathbb{N}_0$ there exists a unique sequence $[n]_{\mathbb{Z}} = \dots d_2 d_1 d_0$ of digits $d_i \in \{0, 1\}$, called the Zeckendorf expansion of n , satisfying

$$n = d_0 F_0 + d_1 F_1 + d_2 F_2 + \dots \quad \text{and} \quad d_i \cdot d_{i+1} = 0 \quad \text{for all } i \geq 0.$$

The last condition means that in the Zeckendorf expansion the digit combination $\dots 11 \dots$ (with two consecutive 1's) is not allowed. Without this condition the Zeckendorf expansion would not be unique as e.g. $6 = F_0 + F_1 + F_2 = F_0 + F_3$.

A number $n \in \mathbb{N}_0$ is said to have even respectively, odd, Zeckendorf expansion if one has $d_0 = 0$, respectively, $d_0 = 1$, in the Zeckendorf expansion $[n]_{\mathbb{Z}} = \dots d_2 d_1 d_0$ of n . For example, the numbers 49 and 50 have even Zeckendorf expansions $[49]_{\mathbb{Z}} = 10100010$ and $[50]_{\mathbb{Z}} = 10100100$ while 51 has odd Zeckendorf expansion $[51]_{\mathbb{Z}} = 10100101$.

For every $i \geq 1$ we let $e(i)$, respectively, $o(i)$ be the i th number in \mathbb{N}_0 whose Zeckendorf expansion is even, respectively, odd. The first few values of the functions e and o are:

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
$e(i)$	0	2	3	5	7	8	10	11	13	15	16	18	20	21	23	...
$o(i)$	1	4	6	9	12	14	17	19	22	25	27	30	33	35	38	...

The number sequences $e(1), e(2), e(3), \dots$ and $o(1), o(2), o(3), \dots$ can be found in the OEIS [9, A022342 and A003622]. By Zeckendorf’s theorem, mentioned above, one has

$$\{e(1), e(2), e(3), \dots\} \uplus \{o(1), o(2), o(3), \dots\} = \mathbb{N}_0.$$

We shall need the following well-known formula, also recorded in the OEIS [9, A003622],

$$(\#18) \quad o(i) = e(e(i) + 1) + 1 \quad \text{for every } i \geq 1.$$

Definition A.5. Define an $\mathbb{N} \times \mathbb{N}$ matrix ∂ with entries from A as follows:

- If $i \in \mathbb{N}$ is a number with odd Zeckendorf expansion ($i = 1, 4, 6, 9, \dots$), then

$$(\#19) \quad \partial_{ij} = \begin{cases} x & \text{if } j = e(i + 1) \\ 0 & \text{otherwise.} \end{cases}$$

- If $i \in \mathbb{N}$ is a number with even Zeckendorf expansion ($i = 2, 3, 5, 7, 8, \dots$), then

$$(\#20) \quad \partial_{ij} = \begin{cases} x & \text{if } j = e(i + 1) \\ y & \text{if } j = e(i + 1) + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the upper left 13×20 corner of ∂ looks like this:

$$\partial = \begin{bmatrix} 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & y & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & y & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

For every $n \geq 1$ let ∂_n denote the $n \times n$ upper left corner of ∂ , for example

$$\partial_1 = [0], \quad \partial_2 = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}, \quad \partial_3 = \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \partial_4 = \begin{bmatrix} 0 & x & 0 & 0 \\ 0 & 0 & x & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Remark A.6. Note that ∂ is both row and column finite, that is, each row and each column in ∂ contains only finitely many (actually at most two) non-zero entries. Furthermore, every entry in ∂ is either 0, x or y .

We view elements in $A^{(\mathbb{N})}$ and A^n as column vectors.

Lemma A.7. *One has $\partial^2 = 0$ and $\partial_n^2 = 0$ for every $n \geq 1$; whence the pairs $(A^{(\mathbb{N})}, \partial)$ and (A^n, ∂_n) are differential A -modules.*

Proof. Note that for each $n \geq 1$ we can write ∂ as a block matrix of the form

$$\partial = \left[\begin{array}{c|c} \partial_n & * \\ \hline 0 & * \end{array} \right] \quad \text{and hence } \partial^2 \text{ has the form } \quad \partial^2 = \left[\begin{array}{c|c} \partial_n^2 & * \\ \hline 0 & * \end{array} \right].$$

Thus, it suffices to argue that $\partial^2 = 0$ holds. Fix $i, j \in \mathbb{N}$. Entry (i, j) in the matrix ∂^2 is the sum $s_{ij} = \sum_{k \in \mathbb{N}} \partial_{ik} \partial_{kj}$, so we must argue that s_{ij} is zero.

If i has odd Zeckendorf expansion, then by (#19) one has $s_{ij} = x \cdot \partial_{e(i+1),j}$, which is zero as $\partial_{e(i+1),j}$ belongs to $\{0, x, y\}$ and $x^2 = xy = 0$ holds in A .

If i has even Zeckendorf expansion, then (#20) yields $s_{ij} = x \cdot \partial_{e(i+1),j} + y \cdot \partial_{e(i+1)+1,j}$. As above, the first term in this sum is zero. As the number i has even Zeckendorf expansion, it has the form $i = e(\ell)$ for some $\ell > 1$. Thus $e(i+1) + 1 = e(e(\ell) + 1) + 1 = o(\ell)$ by (#18), so the number $e(i+1) + 1$ has odd Zeckendorf expansion. Hence $\partial_{e(i+1)+1,j}$ is in $\{0, x\}$ by (#19), and since $yx = 0$ holds in A the second term in the sum s_{ij} is zero as well. □

Lemma A.8. *For every $n \geq 1$ there is a short exact sequence of differential A -modules,*

$$0 \longrightarrow (A^n, \partial_n) \xrightarrow{\iota_n = \begin{bmatrix} 1 \\ 0 \end{bmatrix}} (A^{n+1}, \partial_{n+1}) \xrightarrow{[0 \ 1]} (A, 0) \longrightarrow 0$$

Proof. Since the differential module $(A^{n+1}, \partial_{n+1})$ can be written as

$$(A^{n+1}, \partial_{n+1}) = \left(\begin{array}{c} A^n \\ \oplus \\ A \end{array}, \left[\begin{array}{c|c} \partial_n & * \\ \hline 0 & 0 \end{array} \right] \right),$$

the assertion follows immediately. □

Lemma A.9. *In the abelian category $\text{Diff}(A)$, the (filtered) colimit of the telescope*

$$(\#21) \quad (A, \partial_1) \xrightarrow{\iota_1} (A^2, \partial_2) \xrightarrow{\iota_2} (A^3, \partial_3) \xrightarrow{\iota_3} \dots$$

is the differential module $(A^{(\mathbb{N})}, \partial)$. In symbols: $(A^{(\mathbb{N})}, \partial) = \varinjlim_{n \geq 1} (A^n, \partial_n)$.

Proof. This is evident from the definitions of (A^n, ∂_n) and $(A^{(\mathbb{N})}, \partial)$. □

Proposition A.10. *The differential modules $(A^{(\mathbb{N})}, \partial)$ and (A^n, ∂_n) for $n \geq 1$ belong to ${}^\perp \mathcal{E}$.*

Proof. The class \mathcal{P} in Proposition 3.2(a) is $\mathcal{P} = \{(P, 0) \mid P \in {}_A \text{Prj}\}$. By Lemma A.8 each ι_n is a monomorphism with cokernel $(A, 0)$, which belongs to \mathcal{P} . Recall from Definition A.5 that $\partial_1 = 0$ holds, and therefore (A, ∂_1) is also equal to $(A, 0)$. Hence (#21) is a \mathcal{P} -filtration of $(A^{(\mathbb{N})}, \partial) = \varinjlim_{n \geq 1} (A^n, \partial_n)$ and the assertion follows from Proposition 3.2(a). □

To parse the next result, recall from Neeman [30, Dfn. 1.6.4] the definition of *homotopy colimits* in a triangulated category. Recall from 2.10 that ${}^\perp\mathcal{E}$ is a Frobenius category and its stable category is therefore triangulated by Happel [15, Chap. I§2].

Corollary A.11. *In the stable category of the Frobenius category ${}^\perp\mathcal{E}$, the object $(A^{(\mathbb{N})}, \partial)$ is the homotopy colimit of the sequence (#21). In symbols: $(A^{(\mathbb{N})}, \partial) = \text{hocolim}_{n \geq 1} (A^n, \partial_n)$.*

Proof. We know from Proposition A.10 that (A^n, ∂_n) and $(A^{(\mathbb{N})}, \partial)$ are objects in ${}^\perp\mathcal{E}$. As $(A^{(\mathbb{N})}, \partial)$ is the direct limit of (#21) in $\text{Diff}(A)$ there is a short exact sequence in $\text{Diff}(A)$,

$$(\#22) \quad 0 \longrightarrow \bigoplus_{n \in \mathbb{N}} (A^n, \partial_n) \xrightarrow{\iota} \bigoplus_{n \in \mathbb{N}} (A^n, \partial_n) \longrightarrow (A^{(\mathbb{N})}, \partial) \longrightarrow 0,$$

where σ is the morphism given by

$$\iota(x_1, x_2, x_3, \dots) = (x_1, x_2 - \iota_1(x_1), x_3 - \iota_2(x_2), \dots).$$

As the class ${}^\perp\mathcal{E}$ is closed under coproducts in $\text{Diff}(A)$, the sequence (#22) is a conflation in the Frobenius category ${}^\perp\mathcal{E}$. It now follows from Happel [15, Chap. I§2.7] that there is a distinguished triangle,

$$\bigoplus_{n \in \mathbb{N}} (A^n, \partial_n) \xrightarrow{\iota} \bigoplus_{n \in \mathbb{N}} (A^n, \partial_n) \longrightarrow (A^{(\mathbb{N})}, \partial) \longrightarrow ,$$

in the stable category of ${}^\perp\mathcal{E}$. By definition, see [30, Dfn. 1.6.4], this means that $(A^{(\mathbb{N})}, \partial)$ is the homotopy colimit of the sequence (#21) the stable category of ${}^\perp\mathcal{E}$. □

Lemma A.12. *For $a, b \in A$ one has $ax + by = 0$ if and only if $a \in (x, y)$ and $b \in (x)$.*

Proof. This follows easily from the fact that A is free over \mathbb{k} with basis $\{1, x, y, y^2, y^3, \dots\}$ and from the relations $x^2 = xy = 0$ in A . □

Proposition A.13. *There is a weak equivalence in $\text{Diff}(A)$,*

$$\varphi: (A^{(\mathbb{N})}, \partial) \xrightarrow{\sim} (A, x) \quad \text{given by} \quad (a_1, a_2, a_3, \dots) \longmapsto ya_1;$$

so $(A^{(\mathbb{N})}, \partial)$ is a cofibrant replacement of (A, x) in the projective model structure on $\text{Diff}(A)$.

Proof. For every element (a_1, a_2, \dots) in $A^{(\mathbb{N})}$ one has

$$\varphi\partial(a_1, a_2, \dots) = \varphi(xa_2, xa_3 + ya_4, \dots) = yxa_2 = 0 = xy a_1 = x\varphi(a_1, a_2, \dots),$$

so φ is a morphism of differential A -modules. As $(A^{(\mathbb{N})}, \partial)$ is in ${}^\perp\mathcal{E}$ by Proposition A.10, it is a cofibrant object in the projective model structure on $\text{Diff}(A)$, see 2.10. Below we show

$$(\#23) \quad \text{H}(A^{(\mathbb{N})}, \partial) = A/(x) \oplus 0 \oplus 0 \oplus \dots,$$

and clearly $\text{H}(A, x) = (x, y)/(x)$. Thus, the map $\text{H}(\varphi)$ is the map $A/(x) \rightarrow (x, y)/(x)$ given by multiplication with the element y . Since this is evidently an isomorphism, φ is a weak equivalence by A.2. It remains to prove (#23), which requires a bit more work:

It follows directly from the definition of ∂ , see Definition A.5, that its image is a direct sum, $\text{Im } \partial = \bigoplus_{i \in \mathbb{N}} I_i$, where

$$(\#24) \quad I_i = \begin{cases} (x) & \text{if } i \text{ has odd Zeckendorf expansion} \\ (x, y) & \text{if } i \text{ has even Zeckendorf expansion} \end{cases} \quad (i \in \mathbb{N}).$$

That is,

$$\text{Im } \partial = I_1 \oplus I_2 \oplus I_3 \oplus I_4 \oplus I_5 \oplus \dots = (x) \oplus (x, y) \oplus (x, y) \oplus (x) \oplus (x, y) \oplus \dots.$$

If $(a_1, a_2, a_3, \dots) \in A^{(\mathbb{N})}$ belongs to $\text{Ker } \partial$, then the definition of ∂ shows that one has

$$\begin{cases} xa_{e(i+1)} = 0 & \text{if } i \text{ has odd Zeckendorf expansion} \\ xa_{e(i+1)} + ya_{e(i+1)+1} = 0 & \text{if } i \text{ has even Zeckendorf expansion} \end{cases} \quad (i \in \mathbb{N}).$$

By Lemma A.12 this means that

$$\begin{cases} a_{e(i+1)+1} \in (x) & \text{for every } i \in \mathbb{N} \text{ with even Zeckendorf expansion, and} \\ a_{e(i+1)} \in (x, y) & \text{for every } i \in \mathbb{N}. \end{cases}$$

When i ranges over \mathbb{N} , the values $e(i+1)$ ranges over all natural numbers with even Zeckendorf expansion; thus the last condition above is equivalent to saying that $a_j \in (x, y)$ for every $j \in \mathbb{N}$ with even Zeckendorf expansion. As just mentioned, every $i \in \mathbb{N}$ with even Zeckendorf expansion has the form $i = e(\ell)$ for some $\ell \geq 2$, so the first condition above can be phrased as: $a_{e(e(\ell)+1)+1} \in (x)$ for every $\ell \geq 2$. By using (#18), this is exactly the same as: $a_{o(\ell)} \in (x)$ for every $\ell \geq 2$. When ℓ ranges over numbers ≥ 2 , the values $o(\ell)$ ranges over all natural numbers j with odd Zeckendorf expansion – except for $j = 1$. Thus the first condition above is equivalent to saying that $a_j \in (x)$ for every $j > 1$ with odd Zeckendorf expansion. In conclusion: an element $(a_1, a_2, a_3, \dots) \in A^{(\mathbb{N})}$ belongs to $\text{Ker } \partial$ if and only if

$$\begin{cases} a_j \in (x) & \text{for every } j > 1 \text{ with odd Zeckendorf expansion, and} \\ a_j \in (x, y) & \text{for every } j \in \mathbb{N} \text{ with even Zeckendorf expansion;} \end{cases}$$

note that there is no condition on the element $a_1 \in A$. Consequently, the kernel of ∂ is a direct sum, $\text{Ker } \partial = \bigoplus_{j \in \mathbb{N}} K_j$, where

$$(\#25) \quad K_j = \begin{cases} A & \text{if } j = 1 \\ (x) & \text{if } j > 1 \text{ has odd Zeckendorf expansion} \\ (x, y) & \text{if } j \text{ has even Zeckendorf expansion} \end{cases} \quad (j \in \mathbb{N}).$$

That is,

$$\text{Ker } \partial = K_1 \oplus K_2 \oplus K_3 \oplus K_4 \oplus K_5 \oplus \dots = A \oplus (x, y) \oplus (x, y) \oplus (x) \oplus (x, y) \oplus \dots.$$

Comparing (#24) and (#25) we see that $I_1 = (x) \subset A = K_1$ while $I_i = K_i$ for every $i > 1$. As one has $\text{Ker } \partial / \text{Im } \partial = \bigoplus_{i \in \mathbb{N}} K_i / I_i$, equality (#23) follows. \square

Remark A.14. Since we view elements in $A^{(\mathbb{N})}$ as column vectors, the map φ from Proposition A.13 could be written as the $1 \times \mathbb{N}$ matrix $\varphi = [y \ 0 \ 0 \ \dots]$.

Proof of Theorem A.3. (a): It is immediate from Definition 5.3 that (A, x) is a strictly perfect object in $\text{Diff}(A)$. (In the situation at hand, the category \mathcal{Q} has just one object, so every object in ${}_{\mathcal{Q}, A} \text{Mod} \simeq \text{Diff}(A)$ actually has finite support.)

(b): Note that there is an epimorphism of differential A -modules:

$$F_*(A) = (A^2, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}) \xrightarrow{[1 \ x]} (A, x).$$

By adding this epimorphism to the morphism φ from Proposition A.13 we get, in view of Remark A.14, the epimorphism

$$(L, d) := \left(\begin{array}{c} A^2 \\ \oplus \\ A^{(\mathbb{N})} \end{array}, \left[\begin{array}{c|c} \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} & 0 \\ \hline 0 & \partial \end{array} \right] \right) \xrightarrow{\psi = [1 \ x \ | \ y \ 0 \ 0 \ \dots]} (A, x).$$

As $F_*(A)$ is exact and φ is a weak equivalence, the morphism ψ is a weak equivalence too. Now, consider in $\text{Diff}(A)$ the short exact sequence,

$$0 \longrightarrow \text{Ker } \psi \longrightarrow (L, d) \xrightarrow{\psi} (A, x) \longrightarrow 0.$$

In the projective model structure on $\text{Diff}(A)$ every object is fibrant, see 2.10, and hence the fibrations in this model structure are nothing but epimorphisms, see [20, Dfn. 5.1]. Thus, ψ is a fibration. Since ψ is also a weak equivalence, it follows from [20, Lem. 5.8] that its kernel is exact, that is, $\text{Ker } \psi \in \mathcal{E}$. (Using the same method as in the proof of Proposition A.13, it is not hard to show that $\text{H}(\text{Ker } \psi) = 0$. Some readers may prefer this more direct approach.) Thus, if (A, x) were an object in ${}^\perp \mathcal{E}$, then $\text{Ext}_{\text{Diff}(A)}^1((A, x), \text{Ker } \psi) = 0$ would hold and hence the short exact sequence above would split. But this is impossible; indeed, we now prove that ψ does not have a right inverse (= a section).

Suppose that $\sigma: (A, x) \rightarrow (L, d)$ is a right inverse of ψ in $\text{Diff}(A)$. Every A -linear map $A \rightarrow A^2 \oplus A^{(\mathbb{N})}$; in particular, σ , has the form

$$\sigma(a) = (b'a, b''a, b_1a, b_2a, \dots) \quad , \quad a \in A$$

for some fixed elements b', b'', b_1, b_2, \dots in A (of which only finitely many are non-zero). As σ is a morphism of differential modules, one has $d\sigma = \sigma x$; in particular:

$$(0, b', b_2x, \dots) = d(b', b'', b_1, \dots) = d\sigma(1) = \sigma(x) = (b'x, b''x, b_1x, \dots).$$

By comparing the second coordinates (which is all we need for the argument), we see that $b' = b''x$ holds. As σ is a right inverse of ψ we have:

$$1 = \psi\sigma(1) = \psi(b', b'', b_1, b_2, \dots) = b' + b''x + b_1y = 2b''x + b_1,$$

where the last equality holds by the just established relation $b' = b''x$. But this is impossible since 1 does not belong to the maximal ideal $(x, y) \subset A$.

(c): To prove that (A, x) is neither perfect nor compact in $\mathcal{D}_Q(A) \simeq \text{Ho}(\text{Diff}(A))$, it suffices by Corollary 5.10 to argue that (A, x) is not compact. To this end, let \mathcal{T} be the stable category of the Frobenius category ${}^\perp \mathcal{E}$ and recall from 2.2 that the inclusion functor ${}^\perp \mathcal{E} \rightarrow \text{Diff}(A)$ induces a triangulated equivalence $\mathcal{T} \rightarrow \text{Ho}(\text{Diff}(A))$. By Proposition A.13 the objects (A, x) and $(A^{(\mathbb{N})}, \partial)$ are isomorphic in $\text{Ho}(\text{Diff}(A))$ and by Proposition A.10 the object $(A^{(\mathbb{N})}, \partial)$ is in \mathcal{T} . Hence (A, x) will be a compact object in $\text{Ho}(\text{Diff}(A))$ if and only if $(A^{(\mathbb{N})}, \partial)$ is a compact object in \mathcal{T} . We now show that $(A^{(\mathbb{N})}, \partial)$ is *not* compact in \mathcal{T} .

By Corollary A.11 we have $(A^{(\mathbb{N})}, \partial) = \text{hocolim}_{n \geq 1} (A^n, \partial_n)$ in \mathcal{T} , so if $(A^{(\mathbb{N})}, \partial)$ is compact, then the canonical homomorphism

$$\text{\#26} \quad \varinjlim_{n \geq 1} \text{Hom}_{\mathcal{T}}((A^{(\mathbb{N})}, \partial), (A^n, \partial_n)) \longrightarrow \text{Hom}_{\mathcal{T}}((A^{(\mathbb{N})}, \partial), (A^{(\mathbb{N})}, \partial))$$

is an isomorphism; see Neeman [29, Lem. 2.8] or Rouquier [32, Lem. 3.12]. In particular, the identity morphism on $(A^{(\mathbb{N})}, \partial)$ is in the image of the map (\#26),

which implies that:

(#27) In \mathcal{T} the object $(A^{(\mathbb{N})}, \partial)$ is a direct summand of some (A^n, ∂_n) .

Now consider the ring homomorphism $A \rightarrow A/(x, y) = \mathbb{k}$ and the following composition of functors, where the first functor is the inclusion and the third functor is the canonical one:

$$\perp \mathcal{E} \longrightarrow \text{Diff}(A) \xrightarrow{\mathbb{k} \otimes_A -} \text{Diff}(\mathbb{k}) \longrightarrow \text{Ho}(\text{Diff}(\mathbb{k})).$$

The functor $\mathbb{k} \otimes_A -$ preserves projective objects, and every projective object in $\text{Diff}(\mathbb{k})$ is exact and hence mapped to zero in $\text{Ho}(\text{Diff}(\mathbb{k}))$. It follows that the composite functor above induces a functor $\mathbb{k} \otimes_A -: \mathcal{T} \rightarrow \text{Ho}(\text{Diff}(\mathbb{k}))$. From (#27) we now conclude that

(#28) In $\text{Ho}(\text{Diff}(\mathbb{k}))$ the object $(\mathbb{k}^{(\mathbb{N})}, 0)$ is a direct summand of some $(\mathbb{k}^n, 0)$.

Here we have used that $\mathbb{k} \otimes_A \partial = 0$ and $\mathbb{k} \otimes_A \partial_n = 0$ since the matrices ∂ and ∂_n have entries in the ideal $(x, y) \subset A$, see Definition A.5. Finally, we apply the homology functor $H: \text{Ho}(\text{Diff}(\mathbb{k})) \rightarrow \mathbb{k}\text{Mod}$. From (#28) we now conclude that in the category $\mathbb{k}\text{Mod}$ the module $\mathbb{k}^{(\mathbb{N})}$ is a direct summand of \mathbb{k}^n for some $n \geq 1$. But this is impossible. \square

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REFERENCES

- [1] Luchezar L. Avramov, Ragnar-Olaf Buchweitz, and Srikanth Iyengar, *Class and rank of differential modules*, Invent. Math. **169** (2007), no. 1, 1–35, DOI 10.1007/s00222-007-0041-6. MR2308849
- [2] Luchezar L. Avramov and Hans-Bjørn Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra **71** (1991), no. 2-3, 129–155, DOI 10.1016/0022-4049(91)90144-Q. MR1117631
- [3] Marcel Bökstedt and Amnon Neeman, *Homotopy limits in triangulated categories*, Compositio Math. **86** (1993), no. 2, 209–234. MR1214458
- [4] A. I. Bondal and M. M. Kapranov, *Representable functors, Serre functors, and reconstructions* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 6, 1183–1205, 1337, DOI 10.1070/IM1990v035n03ABEH000716; English transl., Math. USSR-Izv. **35** (1990), no. 3, 519–541. MR1039961
- [5] F. Michel Dekking, *The structure of Zeckendorf expansions*, Integers **21** (2021), Paper No. A6, 10, DOI 10.1080/14697688.2020.1828609. MR4198883
- [6] Ivo Dell’Ambrogio, Greg Stevenson, and Jan Šťovíček, *Gorenstein homological algebra and universal coefficient theorems*, Math. Z. **287** (2017), no. 3-4, 1109–1155, DOI 10.1007/s00209-017-1862-7. MR3719530
- [7] E. Enochs, S. Estrada, and J. R. García-Rozas, *Gorenstein categories and Tate cohomology on projective schemes*, Math. Nachr. **281** (2008), no. 4, 525–540, DOI 10.1002/mana.200510622. MR2404296
- [8] Edgar E. Enochs and Overtoun M. G. Jenda, *Relative homological algebra*, De Gruyter Expositions in Mathematics, vol. 30, Walter de Gruyter & Co., Berlin, 2000, DOI 10.1515/9783110803662. MR1753146
- [9] Steven Finch, *The on-line encyclopedia of integer sequences*, Math. Intelligencer **43** (2021), no. 2, 146–147, DOI 10.1007/s00283-021-10060-2. MR4278491

- [10] J. R. García Rozas, *Covers and envelopes in the category of complexes of modules*, Chapman & Hall/CRC Research Notes in Mathematics, vol. 407, Chapman & Hall/CRC, Boca Raton, FL, 1999. MR1693036
- [11] James Gillespie, *Model structures on exact categories*, J. Pure Appl. Algebra **215** (2011), no. 12, 2892–2902, DOI 10.1016/j.jpaa.2011.04.010. MR2811572
- [12] James Gillespie, *Hereditary abelian model categories*, Bull. Lond. Math. Soc. **48** (2016), no. 6, 895–922, DOI 10.1112/blms/bdw051. MR3608936
- [13] James Gillespie, *Models for homotopy categories of injectives and Gorenstein injectives*, Comm. Algebra **45** (2017), no. 6, 2520–2545, DOI 10.1080/00927872.2016.1233215. MR3594536
- [14] Rüdiger Göbel and Jan Trlifaj, *Approximations and endomorphism algebras of modules*, De Gruyter Expositions in Mathematics, vol. 41, Walter de Gruyter GmbH & Co. KG, Berlin, 2006, DOI 10.1515/9783110199727. MR2251271
- [15] Dieter Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988, DOI 10.1017/CBO9780511629228. MR935124
- [16] Henrik Holm and Peter Jørgensen, *Cotorsion pairs in categories of quiver representations*, Kyoto J. Math. **59** (2019), no. 3, 575–606, DOI 10.1215/21562261-2018-0018. MR3990178
- [17] Henrik Holm and Peter Jørgensen, *Model categories of quiver representations*, Adv. Math. **357** (2019), article no. 106826, 46 pp. MR4013804
- [18] Henrik Holm and Peter Jørgensen, *The Q -shaped derived category of a ring*, J. London Math. Soc. (2), 54 pp., published online DOI: 10.1112/jlms.12662, 2022.
- [19] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999. MR1650134
- [20] Mark Hovey, *Cotorsion pairs, model category structures, and representation theory*, Math. Z. **241** (2002), no. 3, 553–592, DOI 10.1007/s00209-002-0431-9. MR1938704
- [21] Alina Iacob, *Direct sums of exact covers of complexes*, Math. Scand. **98** (2006), no. 2, 161–181, DOI 10.7146/math.scand.a-14989. MR2243700
- [22] Alina Iacob and Srikanth B. Iyengar, *Homological dimensions and regular rings*, J. Algebra **322** (2009), no. 10, 3451–3458, DOI 10.1016/j.jalgebra.2009.08.006. MR2568347
- [23] Takeshi Ishikawa, *On injective modules and flat modules*, J. Math. Soc. Japan **17** (1965), 291–296, DOI 10.2969/jmsj/01730291. MR188272
- [24] Osamu Iyama and Hiroyuki Minamoto, *\mathcal{A} -derived categories*, in preparation.
- [25] Osamu Iyama and Hiroyuki Minamoto, *On a generalization of complexes and their derived categories*, extended abstract of a talk at the 47th Ring and Representation Theory Symposium, Osaka City University, 2014. MR3363946
- [26] G. M. Kelly, *Basic concepts of enriched category theory*, Repr. Theory Appl. Categ. **10** (2005), vi+137. Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714]. MR2177301
- [27] Henning Krause, *Homological theory of representations*, Cambridge Studies in Advanced Mathematics, vol. 195, Cambridge University Press, Cambridge, 2022. MR4327095
- [28] Saunders Mac Lane, *Categories for the working mathematician*, 2nd ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR1712872
- [29] Amnon Neeman, *The Grothendieck duality theorem via Bousfield’s techniques and Brown representability*, J. Amer. Math. Soc. **9** (1996), no. 1, 205–236, DOI 10.1090/S0894-0347-96-00174-9. MR1308405
- [30] Amnon Neeman, *Triangulated categories*, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001, DOI 10.1515/9781400837212. MR1812507
- [31] Jeremy Rickard, *Derived categories and stable equivalence*, J. Pure Appl. Algebra **61** (1989), no. 3, 303–317, DOI 10.1016/0022-4049(89)90081-9. MR1027750
- [32] Raphaël Rouquier, *Dimensions of triangulated categories*, J. K-Theory **1** (2008), no. 2, 193–256, DOI 10.1017/is007011012jkt010. MR2434186
- [33] Manuel Saorín and Jan Štíviček, *On exact categories and applications to triangulated adjoints and model structures*, Adv. Math. **228** (2011), no. 2, 968–1007, DOI 10.1016/j.aim.2011.05.025. MR2822215
- [34] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994, DOI 10.1017/CBO9781139644136. MR1269324

- [35] E. Zeckendorf, *Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas* (French, with English summary), Bull. Soc. Roy. Sci. Liège **41** (1972), 179–182. MR308032

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